Assignment #14

Due on Friday, November 2, 2007

Read Section 7.4 on *The Derivative,* pp. 187–197, in Bressoud.

Read Section 7.6 on *The Chain Rule,* pp. 201–205, in Bressoud.

Do the following problems

1. Recall that a set $U \subseteq \mathbb{R}^n$ is said to be **path connected** iff for any vectors x and y in U, there exists a differentiable path $\sigma\colon [0,1] \to \mathbb{R}^n$ such that $\sigma(0) = x$, $\sigma(1) = y$ and $\sigma(t) \in U$ for all $t \in [0,1]$; i.e., any two elements in U can be connected by a differentiable path whose image is entirely contained in U.

   Suppose that U is an open, path connected subset of $\mathbb{R}^n$. Let $f\colon U \to \mathbb{R}$ be a differentiable scalar field such that $\nabla f(x)$ is the zero vector for all $x \in U$. Prove that f must be constant.

2. Define $f\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \|x\| \quad \text{for all} \;\; x \in \mathbb{R}^n.$$

   (a) Prove that f is differentiable on $\mathbb{R}^n \backslash \{\mathbf{0}\}$, and compute ∇f on that set.
   Suggestion: Observe that $\|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$, compute the partial derivatives of f, and argue that they are continuous on $\mathbb{R}^n \backslash \{\mathbf{0}\}$.

   (b) Let I be and open interval of real numbers, and suppose that $\sigma\colon I \to \mathbb{R}^n$ is a differentiable path satisfying $\sigma(t) \neq \mathbf{0}$ for all $t \in I$. Show that the function $g\colon I \to \mathbb{R}$ defined by $g(t) = \|\sigma(t)\|$ for all $t \in I$ is differentiable on I and compute its derivative.

3. Exercise 6 on page 208 in the text.

4. Let U be an open subset of $\mathbb{R}^n$ and I be an open interval. Suppose that $f\colon U \to \mathbb{R}$ is a differentiable scalar field and $\sigma\colon I \to \mathbb{R}^n$ be a differentiable path whose image lies in U. Suppose also that $\sigma'(t)$ is never the zero vector. Show that if f has a local maximum or a local minimum at some point on the path, then ∇f is perpendicular to the path at that point.

   Suggestion: Consider the real valued function of a single variable $g(t) = f(\sigma(t))$ for all $t \in I$.

5. Let $\sigma\colon [a,b] \to \mathbb{R}^n$ be a differentiable, one–to–one path. Suppose also that $\sigma'(t)$, is never the zero vector. Let $h\colon [c,d] \to [a,b]$ be a one–to–one and onto map such that $h'(t) \neq 0$ for all $t \in [c,d]$. Define

$$\gamma(t) = \sigma(h(t)) \quad \text{for all} \quad t \in [c,d].$$

$\gamma\colon [c,d] \to \mathbb{R}^n$ is a called a *reparametrization* of σ

(a) Show that γ is a differentiable, one–to–one path.

(b) Compute $\gamma'(t)$ and show that it is never the zero vector.

(c) Show that σ and γ have the same image in $\mathbb{R}^n$.