Assignment #15

Due on Wednesday, November 7, 2007

Read Section 3.1 on *The Calculus of Curves,* pp. 53–65, in Bressoud.

Read Section 5.2 on *Line Integrals,* pp. 113–119, in Bressoud.

Do the following problems

1. Let I denote an open interval in $\mathbb{R}$, and $\sigma\colon I \to \mathbb{R}^n$ be a C^1 path. For fixed $a \in I$, define
$$s(t) = \int_a^t \|\sigma'(\tau)\| \,\mathrm{d}\tau \quad \text{for all} \quad t \in I.$$
Show that s is differentiable and compute $s'(t)$ for all $t \in I$.

2. Let σ and s be as defined in the previous problem. Suppose, in addition, that $\sigma'(t)$ is never the zero vector for all t in I. Show that s is a strictly increasing function of t and that it is, therefore, one–to–one.

3. Let σ and s be as defined in problem (1). We can re–parameterize σ by using s as a parameter. We therefore obtain $\sigma(s)$, where s is the *arc length* parameter.

   Differentiate the expression
$$\sigma(s(t)) = \sigma(t)$$
with respect to t using the Chain Rule. Conclude that, if $\sigma'(t)$ is never the zero vector for all t in I, then $\sigma'(s)$ is always a unit vector.

   The vector $\sigma'(s)$ is called the *unit tangent vector* to the path σ.

4. For a and b, positive real numbers, the expression
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
defines an ellipse in the xy–plane $\mathbb{R}^2$.

   Sketch the ellipse, give a parametrization for it, and set up the integral that yields its arc length.

5. Let $\sigma\colon [0,\pi] \to \mathbb{R}^3$ be defined by $\sigma(t) = t\,\widehat{i} + t\sin t\,\widehat{j} + t\cos t\,\widehat{k}$ for all $t \in [0,\pi]$. Compute the arc length of the curve parametrized by σ.