Assignment #15

Due on Wednesday, November 7, 2007

Read Section 3.1 on The Calculus of Curves, pp. 53–65, in Bressoud.
Read Section 5.2 on Line Integrals, pp. 113–119, in Bressoud.

Do the following problems

1. Let $I$ denote an open interval in $\mathbb{R}$, and $\sigma: I \to \mathbb{R}^n$ be a $C^1$ path. For fixed $a \in I$, define
   
   $$ s(t) = \int_a^t \|\sigma'(\tau)\| \, d\tau \quad \text{for all } t \in I. $$

   Show that $s$ is differentiable and compute $s'(t)$ for all $t \in I$.

2. Let $\sigma$ and $s$ be as defined in the previous problem. Suppose, in addition, that $\sigma'(t)$ is never the zero vector for all $t$ in $I$. Show that $s$ is a strictly increasing function of $t$ and that it is, therefore, one–to–one.

3. Let $\sigma$ and $s$ be as defined in problem (1). We can re–parameterize $\sigma$ by using $s$ as a parameter. We therefore obtain $\sigma(s)$, where $s$ is the arc length parameter. Differentiate the expression
   
   $$ \sigma(s(t)) = \sigma(t) $$

   with respect to $t$ using the Chain Rule. Conclude that, if $\sigma'(t)$ is never the zero vector for all $t$ in $I$, then $\sigma'(s)$ is always a unit vector.

   The vector $\sigma'(s)$ is called the unit tangent vector to the path $\sigma$.

4. For $a$ and $b$, positive real numbers, the expression
   
   $$ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 $$

   defines an ellipse in the $xy$–plane $\mathbb{R}^2$.

   Sketch the ellipse, give a parametrization for it, and set up the integral that yields its arc length.

5. Let $\sigma: [0, \pi] \to \mathbb{R}^3$ be defined by $\sigma(t) = t \hat{i} + t \sin t \hat{j} + t \cos t \hat{k}$ for all $t \in [0, \pi]$. Compute the arc length of the curve parametrized by $\sigma$. 