Assignment #5

Due on Monday September 24, 2007

Read Section 7.1 on Limits, pp. 171–178, in Bressoud.

Do the following problems

1. A subset, $U$, of $\mathbb{R}^n$ is said to be **open** if for any $x \in U$ there exists a positive number $r$ such that

   \[ B_r(x) = \{ y \in \mathbb{R}^n \mid \|y - x\| < r \} \]

   is entirely contained in $U$.

   (The empty set, $\emptyset$, is considered to be an open set.)

   (a) Show that if $U_1$ and $U_2$ are open subsets of $\mathbb{R}^n$, then their intersection

   \[ U_1 \cap U_2 = \{ y \in \mathbb{R}^n \mid y \in U_1 \text{ and } y \in U_2 \} \]

   is also open.

   (b) Show that the set

   \[ \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid y = 0 \right\} \]

   is not an open subset of $\mathbb{R}^2$.

2. In problem 2 of Assignment #4 you proved that every linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ must be of the form

   \[ T(v) = w \cdot v \quad \text{for every } v \in \mathbb{R}^n. \]

   Use this fact together with the Cauchy–Schwarz inequality to prove that $T$ is continuous at every point in $\mathbb{R}^n$.

3. A subset, $U$, of $\mathbb{R}^n$ is said to be **convex** if given any two points $x$ and $y$ in $U$, the straight line segment connecting them is entirely contained in $U$; in symbols,

   \[ \{ x + t(y - x) \in \mathbb{R}^n \mid 0 \leq t \leq 1 \} \subseteq U \]
(a) Prove that the ball \( B_r(O) = \{ x \in \mathbb{R}^n \mid \|x\| < R \} \) is a convex subset of \( \mathbb{R}^n \).

(b) Prove that the “punctured unit disc” in \( \mathbb{R}^2 \),
\[
\left\{ \left( \begin{array}{c} x \\ y \end{array} \right) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 1 \right\},
\]
is not a convex set.

4. Let \( x \) and \( y \) denote real numbers.

(a) Starting with the self–evident inequality: \( (|x| − |y|)^2 ≥ 0 \), derive the inequality
\[
|xy| ≤ \frac{1}{2}(x^2 + y^2).
\]

(b) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by
\[
f(x, y) = \begin{cases} 
\frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0),
\end{cases}
\]
Use the inequality derived in the previous part to prove that \( f \) is continuous at the origin.

5. Exercise 10 on page 180 in the text.