Assignment #6

Due on Friday September 28, 2007

Read Section 7.1 on Limits, pp. 171–178, in Bressoud.

Background and Definitions

- Let $U$ denote an open subset of $\mathbb{R}^n$. A function $F: U \to \mathbb{R}^m$ is said to be continuous at $x \in U$ if and only if
  \[
  \lim_{\|y-x\| \to 0} \|F(y) - F(x)\| = 0.
  \]

- If $A \subseteq U$, the image of $A$ under the map $F: U \to \mathbb{R}^m$, denoted by $F(A)$, is defined as the set
  \[
  F(A) = \{ y \in \mathbb{R}^m \mid y = F(x) \text{ for some } x \in A \}.
  \]

- If $B \subseteq \mathbb{R}^m$, the pre-image of $B$ under the map $F: U \to \mathbb{R}^m$, denoted by $F^{-1}(B)$, is defined as the set
  \[
  F^{-1}(B) = \{ x \in U \mid F(x) \in B \}.
  \]
  Note that $F^{-1}(B)$ is always defined even if $F$ does not have an inverse map.

Do the following problems

1. Use the triangle inequality to prove that, for any $x$ and $y$ in $\mathbb{R}^n$,
   \[
   \|\|y\| - \|x\|\| \leq \|y - x\|.
   \]
   Use this inequality to deduce that the function $f: \mathbb{R}^n \to \mathbb{R}$ given by
   \[
   f(x) = \|x\| \quad \text{for all } x \in \mathbb{R}^n
   \]
   is continuous on $\mathbb{R}^n$.

2. Let $f(x, y)$ and $g(x, y)$ denote two functions defined on a open region, $D$, in $\mathbb{R}^2$. Prove that the vector field $F: D \to \mathbb{R}^2$, defined by
   \[
   F \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} f(x, y) \\ g(x, y) \end{array} \right) \quad \text{for all } \left( \begin{array}{c} x \\ y \end{array} \right) \in \mathbb{R}^2,
   \]
   is continuous on $D$ if and only if $f$ and $g$ are both continuous on $D$. 
3. Let $U$ denote an open subset of $\mathbb{R}^n$ and let $F: U \to \mathbb{R}^m$ and $G: U \to \mathbb{R}^m$ be two given functions.

(a) Explain how the sum $F + G$ is defined.
(b) Prove that if both $F$ and $G$ are continuous on $U$, then their sum is also continuous.

(Suggestion: The triangle inequality might come in handy.)

4. In each of the following, given the function $F: U \to \mathbb{R}^m$ and the set $B$, compute the pre–image $F^{-1}(B)$.

(a) $F: \mathbb{R}^2 \to \mathbb{R}^2$, $F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y^2 \\ x^2 - y^2 \end{pmatrix}$, and $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

(b) $f: D' \to \mathbb{R}$,

$$f(x, y) = \frac{1}{\sqrt{1 - x^2 - y^2}}, \quad \text{for } (x, y) \in D'$$

where $D' = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 1\}$ (the punctured unit disc), $B = \{1\}$.

(c) $f: D' \to \mathbb{R}$ is as in part (b), and $B = \{2\}$.

(d) $f: D' \to \mathbb{R}$ is as in part (b), and $B = \{1/2\}$.

5. Compute the image the given sets under the following maps

(a) $\sigma: \mathbb{R} \to \mathbb{R}^2$, $\sigma(t) = (\cos t, \sin t)$ for all $t \in \mathbb{R}$. Compute $\sigma(\mathbb{R})$.

(b) $f: D' \to \mathbb{R}$ and $D'$ are as given in part (b) of the previous problem. Compute $f(D')$. 