Review Problems for Exam 1

1. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the plane given by
$$4x - y - 3z = 12.$$

2. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the line given by the parametric equations
$$\begin{cases} x = -1 + 4t \\ y = -7t \\ z = 2 - t \end{cases}$$

3. Compute the area of the triangle whose vertices in $\mathbb{R}^3$ are the points $(1, 1, 0)$, $(2, 0, 1)$ and $(0, 3, 1)$

4. Let v and w be two vectors in $\mathbb{R}^3$, and let λ be a scalar. Show that the area of the parallelogram determined by the vectors v and $w + \lambda v$ is the same as that determined by v and w.

5. Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$ and $P_{\widehat{u}}(v)$ denote the orthogonal projection of v along the direction of $\widehat{u}$ for any vector $v \in \mathbb{R}^n$. Use the Cauchy–Schwarz inequality to prove that the map
$$v \mapsto P_{\widehat{u}}(v) \quad \text{for all} \quad v \in \mathbb{R}^n$$
is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

6. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by
$$f(x) = \frac{1}{2}\|x\|^2 \quad \text{for all} \quad x \in \mathbb{R}^n.$$
Show that f is differentiable on $\mathbb{R}^n$ and compute the linear map $Df(x)\colon \mathbb{R}^n \to \mathbb{R}$ for all $x \in \mathbb{R}^n$. What is the gradient of f at x for all $x \in \mathbb{R}^n$?

7. A bug finds itself in a plate on the xy–plane whose temperature at any point (x, y) is given by the function
$$T(x, y) = \frac{32}{2 + x^2 - 2x + y^2} \qquad \text{for} \quad (x, y) \in \mathbb{R}^2.$$
Suppose the bug is at the origin and wishes to move in a direction at which the temperature is increasing the fastest. In which direction should the bug move? What is the rate of change of temperature in that direction?

8. Let $g\colon [0,\infty) \to \mathbb{R}$ be a differentiable, real–valued function of a single variable, and let $f(x,y) = g(r)$ where $r = \sqrt{x^2+y^2}$.

   (a) Compute $\dfrac{\partial r}{\partial x}$ in terms of x and r, and $\dfrac{\partial r}{\partial y}$ in terms of y and r.

   (b) Compute ∇f in terms of $g'(r)$, r and the vector $\mathbf{r} = x\widehat{i} + y\widehat{j}$.

9. Let I denote an open interval in $\mathbb{R}$, and suppose that the path $\sigma\colon I \to \mathbb{R}^n$ is differentiable at $t \in I$.

   (a) Show that the linear map $D\sigma(t)\mathbb{R} \to \mathbb{R}^n$ is of the form

   $$D\sigma(t)(h) = hv \quad \text{for all} \quad h \in \mathbb{R},$$

   where the vector $\mathbf{v}(t)$ is obtained from

   $$\mathbf{v} = D\sigma(t)(1);$$

   that is, $\mathbf{v}(t)$ is the image of the real number 1 under the linear transformation $D\sigma(t)$.

   (b) Write $\sigma(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ for all $t \in I$, and

   $$\mathbf{v}(t) = (v_1(t), v_2(t), \ldots, v_n(t))$$

   for all $t \in I$. Show that if $\sigma\colon I \to \mathbb{R}^n$ is differentiable at $t \in I$ and $\mathbf{v} = D\sigma(t)(1)$, then each function $x_j\colon I \to \mathbb{R}$, for $j = 1, 2, \ldots, n$, is differentiable at t, and

   $$x_j'(t) = v_j(t).$$

   Notation: If $\sigma\colon I \to \mathbb{R}^n$ is differentiable at every $t \in I$, the vector valued function $\mathbf{v}\colon I \to \mathbb{R}^n$ given by $\mathbf{v}(t) = D\sigma(t)(1)$ is called the *velocity* of the path σ.

10. Let U denote an open subset of $\mathbb{R}^n$. Suppose that $F\colon U \to \mathbb{R}^m$ and $G\colon U \to \mathbb{R}^m$ is vector valued functions.

    (a) Explain how the scalar product $F \cdot G$ is defined.

    (b) Prove that if both F and G are both differentiable at $x \in U$, the so is $F \cdot G$.

    (c) Define the scalar field $f\colon U \to \mathbb{R}$ by

    $$f(x) = (F \cdot G)(x) \quad \text{for all} \quad x \in U.$$

    If F and G are both differentiable at $x \in U$, compute $\nabla f(x)$.