Assignment #11
Due on Wednesday, November 14, 2007

Read Section 4.2 on An Introduction to Probability, pp. 116–127, in Allman and Rhodes.

Read Section 4.3 on Conditional Probabilities, pp. 130–134, in Allman and Rhodes.

Read Chapter 5 on Modeling Bacterial Mutations in the class lecture notes, starting on page 45, at http://pages.pomona.edu/~ajr04747/

Do the following problems

1. Let $M(t)$ denote number of bacteria in a colony of initial size $N_0$ which develop mutations in the time interval $[0, t]$. It was shown in the lectures that if there are no mutations at time $t = 0$, and if $M(t)$ follows the assumptions of a Poisson process, then the probability of no mutations in the time interval $[0, t]$ is given by

$$P_0(t) = P[M(t) = 0] = e^{-\lambda t}$$

where $\lambda > 0$ is the average number of mutations per unit time, or the mutation rate.

Let $T > 0$ denote the time at which the first mutation occurs.

(a) Explain why $T$ is a random variable. Observe that it is a continuous random variable.

(b) For any $t > 0$, explain why the statement

$$P[T > t] = P[M(t) = 0]$$

is true, and use it to compute

$$F(t) = P[T \leq t].$$

The function $F(t)$, usually denoted by $F_T(t)$, is called the cumulative distribution function, or cdf, of the random variable $T$.

(c) Compute the derivative $f(t) = F'(t)$ of the cdf $F$ obtained in the previous part.

The function $f(t)$, usually denoted by $f_T(t)$, is called the probability density function, or pdf, of the random variable $T$. 

2. Given a continuous random variable $X$ with pdf $f_X$, the expected value of $X$ is defined to be

$$E(X) = \int_{-\infty}^{\infty} xf_X(x)dx.$$ 

Use this formula to compute the expected value of the $T$, where $T$ is the random variable defined in the previous problem; that is, $T > 0$ is the time at which the first mutation occurs for a bacterial colony exposed to a virus at time $t = 0$, assuming that there are no mutations at that time. How does this value relate to the average mutation rate $\lambda$?

3. Given a discrete random variable $X$ with a finite number of possible values $x_1, x_2, x_3, \ldots, x_N$,

the expected value of $X$ is defined to be the sum

$$E(X) = \sum_{i=1}^{N} x_i P[X = x_i].$$

Use this formula to compute the expected value of the numbers appearing on the top face of a fair die. Explain the meaning of this number.

4. Consider the following random experiment: Assume you have a fair die and you toss it until you get a six on the top face, and then you stop. Let $X$ denote the number of tosses you make until you stop.

(a) Explain why $X$ is a discrete random variable. What are the possible values for $X$?

(b) For each value $x$ of $X$, compute $P[X = x]$; this is called the probability mass function, or pmf, of the random variable $X$.

5. Given a discrete random variable $X$ with an infinite number of possible values $x_1, x_2, x_3, \ldots$

the expected value of $X$ is defined to be the infinite series

$$E(X) = \sum_{i=1}^{\infty} x_i P[X = x_i].$$

Use this formula to compute the expected value random variable $X$ of the previous problem; that is, $X$ is the number of times you need to toss a fair die until you get a six on the top face.