Assignment #12

Due on Monday, November 19, 2007

Read Section 4.2 on *An Introduction to Probability,* pp. 116–127, in Allman and Rhodes.

Read Section 4.3 on *Conditional Probabilities,* pp. 130–134, in Allman and Rhodes.

Read Chapter 5 on *Modeling Bacterial Mutations* in the class lecture notes, starting on page 45, at `http://pages.pomona.edu/~ajr04747/`

Do the following problems

1. Consider a hypothetical experiment in which there are only three bacteria in a culture. Suppose that each bacteirum has a small probability p, with $0 < p < 1$, of developing a mutation in a short time interval. Number the bacteria 1, 2 and 3. Use the symbol M to denote the given bacterium mutates in the short time interval, and N to denote that the bacterium did not mutate in that interval.

   (a) List all possible outcomes of the experiment using the symbols M or N, for each of the bacteria 1, 2 and 3, to denote whether a bacterium mutated or not, respectively. This will generate triples made up of the symbols M and N. What is the probability of each outcome?

   (b) Let X denote the number of bacteria that mutate in the short time interval. This defines a discrete random variable. List the possible values for X and give the probability for each of these values. In other words, give the probability mass function for X.

   (c) Compute the expected value and variance of X.

2. Repeat the previous problem in the case of four bacteria, each having a probability p of mutating in a short time interval.

3. Generalize problems 1 and 2 for the case of N bacteria, each having a probability p of mutating in a short time interval.

   For this problem it will be helpful to know that the number of different ways of choosing m bacteria out of N is given by the combinatorial expression

   $$\binom{N}{m} = \frac{N!}{m!(N-m)!},$$

   for $m = 0, 1, 2, \ldots, N$. The symbol $\binom{N}{m}$ is read "N choose m."

   Note: The distribution for X obtained in this problem is called the *binomial* distribution with parameters p and N.

4. In the previous problem you found that the expected number bacteria that mutate in the short time interval is pN. Denote this value by λ, so that $\lambda = pN$. Explore what happens as N gets larger and larger while λ is kept at a fixed value. In particular, compute $\lim\limits_{N\to\infty} P[X = m]$ for any given m. What do you discover?

   Hints:

   i. For this problem it will be helpful to remember that another expression for the exponential function, e^x, is given by the limit
   $$e^x = \lim_{N\to\infty}\left(1 + \frac{x}{N}\right)^N \quad \text{for any real value of } x.$$

   ii. Also, $\dfrac{N!}{N^m(N-m)!} = \left(1 - \dfrac{1}{N}\right)\left(1 - \dfrac{2}{N}\right)\cdots\left(1 - \dfrac{m+1}{N}\right).$

5. *Modeling Survival Time after a Treatment*[1]. Consider a group of people who have received a treatment for a disease such as cancer. Let T denote the *survival time;* that is, T is the number of years a person lives after receiving the treatment. T can be modeled as a continuous random variable with probability density function (pdf) given by
   $$f_T(t) = \begin{cases} \dfrac{1}{\beta}e^{-t/\beta} & \text{for } t \geqslant 0 \\ 0 & \text{for } t < 0, \end{cases}$$
   for some positive constant β. This pdf can be used to compute the probability that, after receiving treatment, a patient will survive between t_1 and t_2 years as follows:
   $$P[t_1 < T < t_2] = \int_{t_1}^{t_2} f_T(t)\, \mathrm{d}t.$$

   (a) Find the expected value of T; that is, compute $E(T) = \displaystyle\int_{-\infty}^{\infty} t f_T(t)\, \mathrm{d}t$.

   (b) The survival function, $S(t)$, is the probability that a randomly selected person will survive for at least t years after receiving treatment. Compute $S(t)$.

   (c) Suppose that a patient has a 70% probability of surviving at least two years. Find β.

[1] Adapted from problem 7 on p. 427 in *Calculus: Single Variable,* Hughes–Hallet *et al.,* Fourth Edition, Wiley, 2005