Assignment #21

Due on Friday, November 21, 2008


Background and Definitions

**Green’s Theorem.** The Fundamental Theorem of Calculus,

\[ \int_M d\omega = \int_{\partial M} \omega, \]

takes the following form in two-dimensional Euclidean space:

Let \( R \) denote a region in \( \mathbb{R}^2 \) bounded by a simple closed curve, \( \partial R \), made up of a finite number of \( C^1 \) paths traversed in the counterclockwise sense. Let \( P \) and \( Q \) denote two \( C^1 \) scalar fields defined on some open set containing \( R \) and its boundary, \( \partial R \). Then,

\[
\int_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \int_{\partial R} P \, dx + Q \, dy. \quad (1)
\]

Do the following problems

1. Apply Green’s Theorem, as expressed in the formula (1), to the functions \( P(x, y) = -y \) and \( Q(x, y) = x \) to derive the formula

\[
\text{area}(R) = \frac{1}{2} \int_{\partial R} -y \, dx + x \, dy. \quad (2)
\]

to compute the area of the region \( R \).

2. Use the formula (2) derived in the previous theorem to compute the area enclosed by the ellipse

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,
\]

where \( a \) and \( b \) are positive real numbers.
3. In Problem 1(b) of Assignment #20, you showed that another form of Fundamental Theorem of Calculus in two dimensions is

\[
\int_R \text{div}(F) \, dx dy = \text{Flux of } F \text{ across } \partial R,
\]

where \( \text{div}(F) = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y} \) is the divergence of the vector field \( F = P \hat{i} + Q \hat{j} \); that is, the flux of \( F \) across the boundary of \( R \) is the double integral of the divergence of \( F \) over the region \( R \). Thus, the Fundamental Theorem of Calculus in \( \mathbb{R}^2 \) takes the form

\[
\int_R \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \, dx dy = \int_{\partial R} F \cdot \hat{n} \, ds, \tag{3}
\]

where \( \hat{n} \) is a unit vector perpendicular to \( \partial R \) and pointing to the outside of \( \partial R \).

Use formula (3) to compute the flux of the field \( F = x \hat{i} + y \hat{j} \) across the square with vertices \((-1, -1), (1, -1), (1, 1) \) and \((-1, 1) \).

4. Let \( f \) and \( g \) be two scalar fields defined on some open subset of \( \mathbb{R}^2 \). Suppose that \( f \) and \( g \) are \( C^1 \) and that \( \nabla g \) is a \( C^1 \) vector field. Show that

\[
\text{div}(f \nabla g) = \nabla f \cdot \nabla g + f \text{div}(\nabla g).
\]

\( \text{div}(\nabla g) \) is called the Laplacian of \( g \) and is usually denoted by \( \Delta g \); thus,

\[
\Delta g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}.
\]

5. Let \( f \) and \( g \) be as in the previous problem. Use formula (3) and the result from the previous problem to show that

\[
\int_R f \Delta g \, dx dy = \int_{\partial R} f \frac{\partial g}{\partial n} \, ds - \int_R \nabla f \cdot \nabla g \, dx dy,
\]

where \( \frac{\partial g}{\partial n} \) denotes the derivative of \( g \) in the direction of \( \hat{n} \), or \( D_{\hat{n}} g \).