Assignment #5
Due on Monday September 22, 2008

Read Section 7.1 on Limits, pp. 171–178, in Bressoud.

Do the following problems

1. A subset, \( U \), of \( \mathbb{R}^n \) is said to be **open** if for any \( x \in U \) there exists a positive number \( r \) such that

\[
B_r(x) = \{ y \in \mathbb{R}^n \mid \|y - x\| < r \}
\]

is entirely contained in \( U \).

(The empty set, \( \emptyset \), is considered to be an open set.)

(a) Show that if \( U_1 \) and \( U_2 \) are open subsets of \( \mathbb{R}^n \), then their intersection

\[
U_1 \cap U_2 = \{ y \in \mathbb{R}^n \mid y \in U_1 \text{ and } y \in U_2 \}
\]

is also open.

(b) Show that the set

\[
\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid y = 0 \right\}
\]

is not an open subset of \( \mathbb{R}^2 \).

2. In problem 2 of Assignment #4 you proved that every linear transformation \( T : \mathbb{R}^n \to \mathbb{R} \) must be of the form

\[
T(v) = w \cdot v \quad \text{for every } v \in \mathbb{R}^n.
\]

Use this fact together with the Cauchy–Schwarz inequality to prove that \( T \) is continuous at every point in \( \mathbb{R}^n \).

3. A subset, \( U \), of \( \mathbb{R}^n \) is said to be **convex** if given any two points \( x \) and \( y \) in \( U \), the straight line segment connecting them is entirely contained in \( U \); in symbols,

\[
\{ x + t(y - x) \in \mathbb{R}^n \mid 0 \leq t \leq 1 \} \subseteq U
\]
(a) Prove that the ball $B_r(O) = \{ x \in \mathbb{R}^n \mid \|x\| < R \}$ is a convex subset of $\mathbb{R}^n$.

(b) Prove that the “punctured unit disc” in $\mathbb{R}^2$,

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 1 \right\},$$

is not a convex set.

4. Let $x$ and $y$ denote real numbers.

(a) Starting with the self–evident inequality: $(|x| - |y|)^2 \geq 0$, derive the inequality

$$|xy| \leq \frac{1}{2}(x^2 + y^2).$$

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0,0), \\
0 & \text{if } (x, y) = (0,0), 
\end{cases}$$

Use the inequality derived in the previous part to prove that $f$ is continuous at the origin.

5. Exercise 10 on page 180 in the text.