Review Problems for Exam 1

1. Compute the (shortest) distance from the point $P(4,0,-7)$ in $\mathbb{R}^3$ to the plane given by
$$4x - y - 3z = 12.$$

2. Compute the (shortest) distance from the point $P(4,0,-7)$ in $\mathbb{R}^3$ to the line given by the parametric equations
$$\begin{cases} x &= -1+4t \\ y &= -7t \\ z &= 2-t \end{cases}$$

3. Compute the area of the triangle whose vertices in $\mathbb{R}^3$ are the points $(1,1,0)$, $(2,0,1)$ and $(0,3,1)$

4. Let v and w be two vectors in $\mathbb{R}^3$, and let λ be a scalar. Show that the area of the parallelogram determined by the vectors v and $w+\lambda v$ is the same as that determined by v and w.

5. Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$ and $P_{\widehat{u}}(v)$ denote the orthogonal projection of v along the direction of $\widehat{u}$ for any vector $v \in \mathbb{R}^n$. Use the Cauchy–Schwarz inequality to prove that the map
$$v \mapsto P_{\widehat{u}}(v) \quad \text{for all} \quad v \in \mathbb{R}^n$$
is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

6. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by
$$f(x) = \frac{1}{2}\|x\|^2 \quad \text{for all} \quad x \in \mathbb{R}^n.$$
Show that f is differentiable on $\mathbb{R}^n$ and compute the linear map $Df(x)\colon \mathbb{R}^n \to \mathbb{R}$ for all $x \in \mathbb{R}^n$. What is the gradient of f at x for all $x \in \mathbb{R}^n$?

7. A bug finds itself in a plate on the xy–plane whose temperature at any point (x,y) is given by the function
$$T(x,y) = \frac{32}{2+x^2-2x+y^2} \qquad \text{for} \quad (x,y) \in \mathbb{R}^2.$$
Suppose the bug is at the origin and wishes to move in a direction at which the temperature is increasing the fastest. In which direction should the bug move? What is the rate of change of temperature in that direction?

8. Let $g\colon [0,\infty) \to \mathbb{R}$ be a differentiable, real–valued function of a single variable, and let $f(x,y) = g(r)$ where $r = \sqrt{x^2+y^2}$.

   (a) Compute $\dfrac{\partial r}{\partial x}$ in terms of x and r, and $\dfrac{\partial r}{\partial y}$ in terms of y and r.

   (b) Compute ∇f in terms of $g'(r)$, r and the vector $\mathbf{r} = x\widehat{i} + y\widehat{j}$.

9. Let D denote an open region in $\mathbb{R}^2$ and $f\colon D \to \mathbb{R}$ denote a scalar field whose second partial derivatives exist in D. Fix $(x,y) \in D$, and define the scalar map

$$S(h,k) = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x,y),$$

   where $|h|$ and $|k|$ are sufficiently small.

   (a) Apply the Mean Value Theorem to obtain an $\overline{x}$ in the interval $(x, x+h)$, or $(x+h, x)$ (depending on whether h is positive or negative, respectively) such that

$$S(h,k) = \left(\frac{\partial f}{\partial x}(\overline{x}, y+k) - \frac{\partial f}{\partial x}(\overline{x}, y)\right) h.$$

   (b) Apply the Mean Value Theorem to obtain a $\overline{y}$ in the interval $(y, y+k)$, or $(y+k, y)$ (depending on whether k is positive or negative, respectively) such that

$$S(h,k) = \frac{\partial^2 f}{\partial y \partial x}(\overline{x}, \overline{y}) hk.$$

10. *(Continuation of Problem 9.)*

   (c) The function f is said to be of class C^2 if all its second partial derivatives are continuous on D.

   Show that if f is of class C^2, then

$$\lim_{(h,k)\to(0,0)} \frac{S(h,k)}{hk} = \frac{\partial^2 f}{\partial y \partial x}(x,y).$$

   (d) Deduce that if f is of class C^2, then

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y);$$

   that is, the *mixed* second partial derivatives are the same for C^2 maps.