Review Problems for Exam 1

1. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the plane given by
$$4x - y - 3z = 12.$$

   ***Solution**:* The point $P_o(3, 0, 0)$ is in the plane. Let $w = \overrightarrow{P_oP} = \begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix}$.

   The vector $n = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$ is orthogonal to the plane. To find the shortest distance, d, from P to the plane, we compute the norm of the orthogonal projection of w onto n; that is,
$$d = \|\text{Proj}_{\widehat{n}}(w)\|,$$
   where
$$\widehat{n} = \frac{1}{\sqrt{26}} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix},$$
   a unit vector in the direction of n, and
$$\text{Proj}_{\widehat{n}}(w) = (w \cdot \widehat{n})\widehat{n}.$$
   It then follows that
$$d = |w \cdot \widehat{n}|,$$
   where $w \cdot \widehat{n} = \dfrac{1}{\sqrt{26}}(4 + 21) = \dfrac{25}{\sqrt{26}}$. Hence, $d = \dfrac{25\sqrt{26}}{26} \approx 4.9.$ $\square$

2. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the line given by the parametric equations
$$\begin{cases} x = -1 + 4t \\ y = -7t \\ z = 2 - t \end{cases}$$

***Solution**:* The point $P_o(-1,0,2)$ is on the line. The vector $v = \begin{pmatrix} 4 \\ -7 \\ -1 \end{pmatrix}$ gives the direction of the line. Put $w = \overrightarrow{P_oP} = \begin{pmatrix} 5 \\ 0 \\ -9 \end{pmatrix}$. The vectors v and w determine a parallelogram whose area is the norm of v times the shortest distance, d, from P to the line determined by v at P_o. We then have that

$$\text{area} v, w = \|v\| d,$$

from which we get that

$$d = \frac{\text{area}\{v, w\}}{\|v\|}.$$

On the other hand,

$$\text{area}\{v, w\} = \|v \times w\|,$$

where

$$v \times w = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 4 & -7 & -1 \\ 5 & 0 & -9 \end{vmatrix} = 63\widehat{i} + 31\widehat{j} - 35\widehat{k}.$$

Thus, $\|v \times w\| = \sqrt{(63)^2 + (31)^2 + (35)^2} = \sqrt{6155}$ and therefore

$$d = \frac{\sqrt{6155}}{\sqrt{66}} \approx 9.7.$$

□

3. Compute the area of the triangle whose vertices in $\mathbb{R}^3$ are the points $(1,1,0)$, $(2,0,1)$ and $(0,3,1)$

***Solution**:* Label the points $P_o(1,1,0)$, $P_1(2,0,1)$ and $P_2(0,3,1)$ and define the vectors

$$v = \overrightarrow{P_oP_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad w = \overrightarrow{P_oP_2} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

The area of the triangle determined by the points P_o, P_1 and P_2 is then half of the area of the parallelogram determined by the vectors v and w. Thus,

$$\text{area}(\triangle P_oP_1P_2) = \frac{1}{2}\|v \times w\|,$$

where

$$v \times w = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 1 & -1 & 1 \\ -1 & 2 & 1 \end{vmatrix} = -3\widehat{i} - 2\widehat{j} + \widehat{k}.$$

Consequently, $\text{area}(\triangle P_o P_1 P_2) = \dfrac{1}{2}\sqrt{9+4+1} = \dfrac{\sqrt{14}}{2} \approx 1.87.$

□

4. Let v and w be two vectors in $\mathbb{R}^3$, and let λ be a scalar. Show that the area of the parallelogram determined by the vectors v and $w + \lambda v$ is the same as that determined by v and w.

***Solution**:* The area of the parallelogram determined by v and $w+\lambda v$ is

$$\text{area}\{v, w + \lambda v\} = \|v \times (w + \lambda v)\|,$$

where

$$v \times (w + \lambda v) = v \times w + \lambda v \times v = v \times w.$$

Consequently, $\text{area}\{v, w + \lambda v\} = \|v \times w\| = \text{area}\{v, w\}$. □

5. Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$ and $P_{\widehat{u}}(v)$ denote the orthogonal projection of v along the direction of $\widehat{u}$ for any vector $v \in \mathbb{R}^n$. Use the Cauchy–Schwarz inequality to prove that the map

$$v \mapsto P_{\widehat{u}}(v) \quad \text{for all} \quad v \in \mathbb{R}^n$$

is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

***Solution**:* $P_{\widehat{u}}(v) = (v \cdot \widehat{u}) widehatu$ for all $v \in \mathbb{R}^n$. Consequently, for any $w, v \in \mathbb{R}^n$,

$$\begin{aligned} P_{\widehat{u}}(w) - P_{\widehat{u}}(v) &= (w \cdot \widehat{u})\widehat{u} - (v \cdot \widehat{u})\widehat{u} \\ &= (w \cdot \widehat{u} - v \cdot \widehat{u})\widehat{u} \\ &= [(w - v) \cdot \widehat{u}]\widehat{u}. \end{aligned}$$

It then follows that

$$\|P_{\widehat{u}}(w) - P_{\widehat{u}}(v)\| = |(w - v) \cdot \widehat{u}|,$$

since $\|\widehat{u}\| = 1$. Hence, by the Cauchy–Schwarz inequality,

$$\|P_{\widehat{u}}(w) - P_{\widehat{u}}(v)\| \leqslant \|w - v\|.$$

Applying the Squeeze Theorem we then get that

$$\lim_{\|w-v\|\to 0} \|P_{\widehat{u}}(w) - P_{\widehat{u}}(v)\| = 0,$$

which shows that $P_{\widehat{u}}$ is continuous at every $v \in V$. □

6. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \frac{1}{2}\|x\|^2 \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Show that f is differentiable on $\mathbb{R}^n$ and compute the linear map $Df(x)\colon \mathbb{R}^n \to \mathbb{R}$ for all $x \in \mathbb{R}^n$. What is the gradient of f at x for all $x \in \mathbb{R}^n$?

***Solution**:* Let u and w be any vector in $\mathbb{R}^n$ and consider

$$\begin{aligned} f(u+w) &= \tfrac{1}{2}\|u+w\|^2 \\ &= \tfrac{1}{2}(u+w)\cdot(u+w) \\ &= \tfrac{1}{2}u\cdot u + u\cdot w + \tfrac{1}{2}w\cdot w \\ &= \tfrac{1}{2}\|u\|^2 + u\cdot w + \tfrac{1}{2}\|w\|^2. \end{aligned}$$

Thus,

$$f(u+w) - f(u) - u\cdot w = \frac{1}{2}\|w\|^2.$$

Consequently,

$$\frac{|f(u+w) - f(u) - u\cdot w|}{\|w\|} = \frac{1}{2}\|w\|,$$

from which we get that

$$\lim_{\|w\|\to 0} \frac{|f(u+w) - f(u) - u\cdot w|}{\|w\|} = 0,$$

and therefore f is differentiable at u with derivative map $Df(u)$ given by

$$Df(u)w = u\cdot w \quad \text{for all } w \in \mathbb{R}^n.$$

Hence, $\nabla f(u) = u$ for all $u \in \mathbb{R}^n$. □

7. A bug finds itself in a plate on the xy–plane whose temperature at any point (x, y) is given by the function

$$T(x,y) = \frac{32}{2 + x^2 - 2x + y^2} \qquad \text{for} \quad (x,y) \in \mathbb{R}^2.$$

Suppose the bug is at the origin and wishes to move in a direction at which the temperature is increasing the fastest. In which direction should the bug move? What is the rate of change of temperature in that direction?

Solution: The direction of maximum increase at $(0,0)$ is the direction of the gradient of T at that point, $\nabla T(0,0)$, where

$$\nabla T(x,y) = \frac{\partial T}{\partial x}(x,y)\widehat{\imath} + \frac{\partial T}{\partial y}(x,y)\widehat{\jmath}.$$

Computing the partial derivatives we obtain that

$$\frac{\partial T}{\partial x}(x,y) = -64\frac{x-1}{(2+x^2-2x+y^2)^2} \qquad \text{for} \quad (x,y)\in\mathbb{R}^2,$$

and

$$\frac{\partial T}{\partial y}(x,y) = -64\frac{y}{(2+x^2-2x+y^2)^2} \qquad \text{for} \quad (x,y)\in\mathbb{R}^2.$$

It then follows that

$$\nabla T(0,0) = 16\widehat{\imath}.$$

Thus, the bug needs to move in the direction of the vector $\widehat{\imath}$ for the temperature to increase the fastest. The rated of change of temperature in that direction is the magnitude of the gradient at $(0,0)$; namely, 16.

□

8. Let $g\colon [0,\infty)\to\mathbb{R}$ be a differentiable, real–valued function of a single variable, and let $f(x,y) = g(r)$ where $r = \sqrt{x^2+y^2}$.

   (a) Compute $\dfrac{\partial r}{\partial x}$ in terms of x and r, and $\dfrac{\partial r}{\partial y}$ in terms of y and r.

   Solution: Take the partial derivative of $r^2 = x^2+y^2$ on both sides with respect to x to obtain

   $$\frac{\partial(r^2)}{\partial x} = 2x.$$

   Applying the chain rule on the left–hand side we get

   $$2r\frac{\partial r}{\partial x} = 2x,$$

   which leads to

   $$\frac{\partial r}{\partial x} = \frac{x}{r}.$$

   Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$. □

(b) Compute ∇f in terms of $g'(r)$, r and the vector $\mathbf{r} = x\widehat{i} + y\widehat{j}$.

Solution: Take the partial derivative of $f(x,y) = g(r)$ on both sides with respect to x and apply the Chain Rule to obtain

$$\frac{\partial f}{\partial x} = g'(r)\frac{\partial r}{\partial x} = g'(r)\frac{x}{r}.$$

Similarly, $\dfrac{\partial f}{\partial y} = g'(r)\dfrac{y}{r}$.
It then follows that

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial x}\widehat{i} + \frac{\partial f}{\partial y}\widehat{j} \\ &= g'(r)\frac{x}{r}\widehat{i} + g'(r)\frac{y}{r}\widehat{j} \\ &= \frac{g'(r)}{r}(x\widehat{i} + y\widehat{j}) \\ &= \frac{g'(r)}{r}\mathbf{r}.\end{aligned}$$

□

9. Let D denote an open region in $\mathbb{R}^2$ and $f\colon D \to \mathbb{R}$ denote a scalar field whose second partial derivatives exist in D. Fix $(x,y) \in D$, and define the scalar map

$$S(h,k) = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x,y),$$

where $|h|$ and $|k|$ are sufficiently small.

(a) Apply the Mean Value Theorem to obtain an $\overline{x}$ in the interval $(x, x+h)$, or $(x+h, x)$ (depending on whether h is positive or negative, respectively) such that

$$S(h,k) = \left(\frac{\partial f}{\partial x}(\overline{x}, y+k) - \frac{\partial f}{\partial x}(\overline{x}, y)\right) h.$$

Solution: For fixed y, let $g(x) = f(x, y+k) - f(x)$. It then follows that g is differentiable with

$$g'(x) = \frac{\partial f}{\partial x}(x, y+k) - \frac{\partial f}{\partial x}(x,y).$$

Also,

$$S(h,k) = g(x+h) - g(x).$$

By the Mean Value Theorem, there exists $\overline{x}$ between x and $x+h$, such that

$$g(x+h) - g(x) = g'(\overline{x})h.$$

It then follows that

$$S(h,k) = \left(\frac{\partial f}{\partial x}(\overline{x}, y+k) - \frac{\partial f}{\partial x}(\overline{x}, y)\right) h.$$

□

(b) Apply the Mean Value Theorem to obtain a $\overline{y}$ in the interval $(y, y+k)$, or $(y+k, y)$ (depending on whether k is positive or negative, respectively) such that

$$S(h,k) = \frac{\partial^2 f}{\partial y \partial x}(\overline{x}, \overline{y})hk.$$

Solution: Define $g(y) = \dfrac{\partial f}{\partial x}(\overline{x}, y)$. Then, g is differentiable with

$$g'(y) = \frac{\partial^2 f}{\partial y \partial x}(\overline{x}, y).$$

By the Mean Value Theorem, there exists $\overline{y}$ between y and $y+k$ such that

$$g(y+k) - g(y) = g'(\overline{y})k.$$

It then follows that

$$\frac{\partial f}{\partial x}(\overline{x}, y+k) - \frac{\partial f}{\partial x}(\overline{x}, y) = \frac{\partial^2 f}{\partial y \partial x}(\overline{x}, \overline{y})k.$$

Consequently, from the previous part,

$$S(h,k) = \left(\frac{\partial f}{\partial x}(\overline{x}, y+k) - \frac{\partial f}{\partial x}(\overline{x}, y)\right) h = \frac{\partial^2 f}{\partial y \partial x}(\overline{x}, \overline{y})kh,$$

which was to be shown. □

10. *(Continuation of Problem 9.)*

(c) The function f is said to be of class C^2 if all its second partial derivatives are continuous on D.

Show that if f is of class C^2, then

$$\lim_{(h,k)\to(0,0)} \frac{S(h,k)}{hk} = \frac{\partial^2 f}{\partial y \partial x}(x,y).$$

Solution: From part (b) of Problem 9 we get that, for $h \neq 0$ and $k \neq 0$,

$$\frac{S(h,k)}{hk} = \frac{\partial^2 f}{\partial y \partial x}(\overline{x}, \overline{y}),$$

where $\overline{x}$ is between x and $x+h$, and $\overline{y}$ is between y and $y+k$. It then follows that $\overline{x} \to x$ and $\overline{y} \to y$ as $(h,k) \to (0,0)$. Consequently, since the second partial derivatives of f are continuous,

$$\lim_{(h,k)\to(0,0)} \frac{S(h,k)}{hk} = \frac{\partial^2 f}{\partial y \partial x}(x,y).$$

□

(d) Deduce that if f is of class C^2, then

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y);$$

that is, the *mixed* second partial derivatives are the same for C^2 maps.

Solution: An argument similar to that in Problem 9(a), with $g(y) = f(x+h,y) - f(x,y)$, leads to

$$S(h,k) = \left(\frac{\partial f}{\partial y}(x+h, \overline{y}) - \frac{\partial f}{\partial y}(x, \overline{y}) \right) k,$$

for some $\overline{y}$ between y and $y+k$. Consequently, by the Mean Value Theorem again, there exists $\overline{x}$ between x and $x+h$ such that

$$S(h,k) = \frac{\partial^2 f}{\partial x \partial y}(\overline{x}, \overline{y}) hk.$$

Hence, the argument used in the previous part yields that, if the second partial derivatives of f are continuous,

$$\lim_{(h,k)\to(0,0)} \frac{S(h,k)}{hk} = \frac{\partial^2 f}{\partial x \partial y}(x,y).$$

It then follows that, if f is of class C^2,

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y).$$

□