Review Problems for Exam 1

1. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the plane given by

$$4x - y - 3z = 12.$$

**Solution:** The point $P_o(3, 0, 0)$ is in the plane. Let $w = \overrightarrow{P_oP} = \begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix}$.

The vector $n = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$ is orthogonal to the plane. To find the shortest distance, $d$, from $P$ to the plane, we compute the norm of the orthogonal projection of $w$ onto $n$; that is,

$$d = \|\text{Proj}_n(w)\|,$$

where

$$\hat{n} = \frac{1}{\sqrt{26}} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix},$$

a unit vector in the direction of $n$, and

$$\text{Proj}_n(w) = (w \cdot \hat{n})\hat{n}.$$ 

It then follows that

$$d = |w \cdot \hat{n}|,$$

where $w \cdot \hat{n} = \frac{1}{\sqrt{26}}(4 + 21) = \frac{25}{\sqrt{26}}$. Hence, $d = \frac{25\sqrt{26}}{26} \approx 4.9$. □

2. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the line given by the parametric equations

$$\begin{cases}
  x = -1 + 4t \\
  y = -7t \\
  z = 2 - t
\end{cases}$$
**Solution:** The point \( P_o(-1,0,2) \) is on the line. The vector \( v = \begin{pmatrix} 4 \\ -7 \\ -1 \end{pmatrix} \) gives the direction of the line. Put \( w = \overrightarrow{P_oP} = \begin{pmatrix} 5 \\ 0 \\ -9 \end{pmatrix} \). The vectors \( v \) and \( w \) determine a parallelogram whose area is the norm of \( v \) times the shortest distance, \( d \), from \( P \) to the line determined by \( v \) at \( P_o \). We then have that

\[
\text{area} \{ v, w \} = \|v\| d,
\]

from which we get that

\[
d = \frac{\text{area}\{v, w\}}{\|v\|}.
\]

On the other hand,

\[
\text{area}\{v, w\} = \|v \times w\|
\]

where

\[
v \times w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -7 & -1 \\ 5 & 0 & -9 \end{vmatrix} = 63\hat{i} + 31\hat{j} - 35\hat{k}.
\]

Thus, \( \|v \times w\| = \sqrt{(63)^2 + (31)^2 + (35)^2} = \sqrt{6155} \) and therefore

\[
d = \frac{\sqrt{6155}}{\sqrt{66}} \approx 9.7.
\]

3. Compute the area of the triangle whose vertices in \( \mathbb{R}^3 \) are the points \((1,1,0), (2,0,1)\) and \((0,3,1)\)

**Solution:** Label the points \( P_o(1,1,0), P_1(2,0,1) \) and \( P_2(0,3,1) \) and define the vectors

\[
v = \overrightarrow{P_oP_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad w = \overrightarrow{P_oP_2} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.
\]

The area of the triangle determined by the points \( P_o, P_1 \) and \( P_2 \) is then half of the area of the parallelogram determined by the vectors \( v \) and \( w \). Thus,

\[
\text{area}(\triangle P_oP_1P_2) = \frac{1}{2}\|v \times w\|,
\]
where

\[ v \times w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ -1 & 2 & 1 \end{vmatrix} = -3\hat{i} - 2\hat{j} + \hat{k}. \]

Consequently, \( \text{area}(\triangle P_0P_1P_2) = \frac{1}{2}\sqrt{9 + 4 + 1} = \frac{\sqrt{14}}{2} \approx 1.87. \) □

4. Let \( v \) and \( w \) be two vectors in \( \mathbb{R}^3 \), and let \( \lambda \) be a scalar. Show that the area of the parallelogram determined by the vectors \( v \) and \( w + \lambda v \) is the same as that determined by \( v \) and \( w \).

**Solution:** The area of the parallelogram determined by \( v \) and \( w + \lambda v \) is

\[ \text{area}\{v, w + \lambda v\} = \|v \times (w + \lambda v)\|, \]

where

\[ v \times (w + \lambda v) = v \times w + \lambda v \times v = v \times w. \]

Consequently, \( \text{area}\{v, w + \lambda v\} = \|v \times w\| = \text{area}\{v, w\}. \) □

5. Let \( \hat{u} \) denote a unit vector in \( \mathbb{R}^n \) and \( P_{\hat{u}}(v) \) denote the orthogonal projection of \( v \) along the direction of \( \hat{u} \) for any vector \( v \in \mathbb{R}^n \). Use the Cauchy–Schwarz inequality to prove that the map

\[ v \mapsto P_{\hat{u}}(v) \quad \text{for all } v \in \mathbb{R}^n \]

is a continuous map from \( \mathbb{R}^n \) to \( \mathbb{R}^n \).

**Solution:** \( P_{\hat{u}}(v) = (v \cdot \hat{u})\hat{u} \) for all \( v \in \mathbb{R}^n \). Consequently, for any \( w, v \in \mathbb{R}^n \),

\[
P_{\hat{u}}(w) - P_{\hat{u}}(v) = (w \cdot \hat{u})\hat{u} - (v \cdot \hat{u})\hat{u}
= (w \cdot \hat{u} - v \cdot \hat{u})\hat{u}
= [(w - v) \cdot \hat{u}]\hat{u}.
\]

It then follows that

\[
\|P_{\hat{u}}(w) - P_{\hat{u}}(v)\| = |(w - v) \cdot \hat{u}|,
\]

since \( \|\hat{u}\| = 1 \). Hence, by the Cauchy–Schwarz inequality,

\[
\|P_{\hat{u}}(w) - P_{\hat{u}}(v)\| \leq \|w - v\|.
\]
Applying the Squeeze Theorem we then get that
\[ \lim_{\|w-v\|\to 0} \|P_\hat{u}(w) - P_\hat{u}(v)\| = 0, \]
which shows that \(P_\hat{u}\) is continuous at every \(v \in V\). \(\square\)

6. Define the scalar field \(f : \mathbb{R}^n \to \mathbb{R}\) by
\[ f(x) = \frac{1}{2} \|x\|^2 \quad \text{for all } x \in \mathbb{R}^n. \]
Show that \(f\) is differentiable on \(\mathbb{R}^n\) and compute the linear map \(Df(x) : \mathbb{R}^n \to \mathbb{R}\) for all \(x \in \mathbb{R}^n\). What is the gradient of \(f\) at \(x\) for all \(x \in \mathbb{R}^n\)?

\textbf{Solution:}\ Let \(u\) and \(w\) be any vector in \(\mathbb{R}^n\) and consider
\[
\begin{align*}
f(u + w) &= \frac{1}{2} \|u + w\|^2 \\
&= \frac{1}{2} (u + w) \cdot (u + w) \\
&= \frac{1}{2} u \cdot u + u \cdot w + \frac{1}{2} w \cdot w \\
&= \frac{1}{2} \|u\|^2 + u \cdot w + \frac{1}{2} \|w\|^2.
\end{align*}
\]
Thus,
\[
f(u + w) - f(u) - u \cdot w = \frac{1}{2} \|w\|^2.
\]
Consequently,
\[
\left| \frac{f(u + w) - f(u) - u \cdot w}{\|w\|} \right| = \frac{1}{2} \|w\|,
\]
from which we get that
\[
\lim_{\|w\| \to 0} \frac{|f(u + w) - f(u) - u \cdot w|}{\|w\|} = 0,
\]
and therefore \(f\) is differentiable at \(u\) with derivative map \(Df(u)\) given by
\[
Df(u)w = u \cdot w \quad \text{for all } w \in \mathbb{R}^n.
\]
Hence, \(\nabla f(u) = u\) for all \(u \in \mathbb{R}^n\). \(\square\)

7. A bug finds itself in a plate on the \(xy\)-plane whose temperature at any point \((x, y)\) is given by the function
\[
T(x, y) = \frac{32}{2 + x^2 - 2x + y^2} \quad \text{for } (x, y) \in \mathbb{R}^2.
\]
Suppose the bug is at the origin and wishes to move in a direction at which the temperature is increasing the fastest. In which direction should the bug move? What is the rate of change of temperature in that direction?
Solution: The direction of maximum increase at \((0, 0)\) is the direction of the gradient of \(T\) at that point, \(\nabla T(0, 0)\), where
\[
\nabla T(x, y) = \frac{\partial T}{\partial x}(x, y)\hat{i} + \frac{\partial T}{\partial y}(x, y)\hat{j}.
\]
Computing the partial derivatives we obtain that
\[
\frac{\partial T}{\partial x}(x, y) = -64\frac{x - 1}{(2 + x^2 - 2x + y^2)^2} \quad \text{for } (x, y) \in \mathbb{R}^2,
\]
and
\[
\frac{\partial T}{\partial y}(x, y) = -64\frac{y}{(2 + x^2 - 2x + y^2)^2} \quad \text{for } (x, y) \in \mathbb{R}^2.
\]
It then follows that
\[
\nabla T(0, 0) = 16\hat{i}.
\]
Thus, the bug needs to move in the direction of the vector \(\hat{i}\) for the temperature to increase the fastest. The rate of change of temperature in that direction is the magnitude of the gradient at \((0, 0)\); namely, 16.

8. Let \(g: [0, \infty) \to \mathbb{R}\) be a differentiable, real–valued function of a single variable, and let \(f(x, y) = g(r)\) where \(r = \sqrt{x^2 + y^2}\).

(a) Compute \(\frac{\partial r}{\partial x}\) in terms of \(x\) and \(r\), and \(\frac{\partial r}{\partial y}\) in terms of \(y\) and \(r\).

Solution: Take the partial derivative of \(r^2 = x^2 + y^2\) on both sides with respect to \(x\) to obtain
\[
\frac{\partial (r^2)}{\partial x} = 2x.
\]
Applying the chain rule on the left–hand side we get
\[
2r \frac{\partial r}{\partial x} = 2x,
\]
which leads to
\[
\frac{\partial r}{\partial x} = \frac{x}{r}.
\]
Similarly, \(\frac{\partial r}{\partial y} = \frac{y}{r}\).
(b) Compute $\nabla f$ in terms of $g'(r)$, $r$ and the vector $\mathbf{r} = x\hat{i} + y\hat{j}$.

**Solution:** Take the partial derivative of $f(x, y) = g(r)$ on both sides with respect to $x$ and apply the Chain Rule to obtain

$$\frac{\partial f}{\partial x} = g'(r) \frac{\partial r}{\partial x} = g'(r) \frac{x}{r}.$$  

Similarly, $\frac{\partial f}{\partial y} = g'(r) \frac{y}{r}$.

It then follows that

$$\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} = g'(r) \frac{x\hat{i} + y\hat{j}}{r}.$$  

$$= \frac{g'(r)}{r} (x\hat{i} + y\hat{j}).$$  

$$= \frac{g'(r)}{r} \mathbf{r}.$$  

\[\square\]

9. Let $D$ denote an open region in $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ denote a scalar field whose second partial derivatives exist in $D$. Fix $(x, y) \in D$, and define the scalar map

$$S(h, k) = f(x + h, y + k) - f(x + h, y) - f(x, y + k) + f(x, y),$$

where $|h|$ and $|k|$ are sufficiently small.

(a) Apply the Mean Value Theorem to obtain an $\bar{x}$ in the interval $(x, x + h)$, or $(x + h, x)$ (depending on whether $h$ is positive or negative, respectively) such that

$$S(h, k) = \left( \frac{\partial f}{\partial x}(\bar{x}, y + k) - \frac{\partial f}{\partial x}(\bar{x}, y) \right) h.$$  

**Solution:** For fixed $y$, let $g(x) = f(x, y + k) - f(x)$. It then follows that $g$ is differentiable with

$$g'(x) = \frac{\partial f}{\partial x}(x, y + k) - \frac{\partial f}{\partial x}(x, y).$$  

Also,

$$S(h, k) = g(x + h) - g(x).$$
By the Mean Value Theorem, there exists $\bar{x}$ between $x$ and $x+h$, such that

$$g(x + h) - g(x) = g'(\bar{x})h.$$ 

It then follows that

$$S(h,k) = \left( \frac{\partial f}{\partial x}(\bar{x}, y + k) - \frac{\partial f}{\partial x}(\bar{x}, y) \right) h.$$

\[\square\]

(b) Apply the Mean Value Theorem to obtain a $\bar{y}$ in the interval $(y, y + k)$, or $(y + k, y)$ (depending on whether $k$ is positive or negative, respectively) such that

$$S(h, k) = \frac{\partial^2 f}{\partial y \partial x}(\bar{x}, \bar{y})hk.$$ 

**Solution:** Define $g(y) = \frac{\partial f}{\partial x}(\bar{x}, y)$. Then, $g$ is differentiable with

$$g'(y) = \frac{\partial^2 f}{\partial y \partial x}(\bar{x}, y).$$

By the Mean Value Theorem, there exists $\bar{y}$ between $y$ and $y + k$ such that

$$g(y + k) - g(y) = g'(\bar{y})k.$$ 

It then follows that

$$\frac{\partial f}{\partial x}(\bar{x}, y + k) - \frac{\partial f}{\partial x}(\bar{x}, y) = \frac{\partial^2 f}{\partial y \partial x}(\bar{x}, \bar{y})k.$$ 

Consequently, from the previous part,

$$S(h, k) = \left( \frac{\partial f}{\partial x}(\bar{x}, y + k) - \frac{\partial f}{\partial x}(\bar{x}, y) \right) h = \frac{\partial^2 f}{\partial y \partial x}(\bar{x}, \bar{y})kh,$$

which was to be shown. \[\square\]

10. *(Continuation of Problem 9.)*

(c) The function $f$ is said to be of class $C^2$ if all its second partial derivatives are continuous on $D$.

Show that if $f$ is of class $C^2$, then

$$\lim_{(h,k)\to(0,0)} \frac{S(h,k)}{hk} = \frac{\partial^2 f}{\partial y \partial x}(x, y).$$
Solution: From part (b) of Problem 9 we get that, for \( h \neq 0 \) and \( k \neq 0 \),

\[
\frac{S(h, k)}{hk} = \frac{\partial^2 f}{\partial y \partial x}(\bar{x}, \bar{y}),
\]

where \( \bar{x} \) is between \( x \) and \( x + h \), and \( \bar{y} \) is between \( y \) and \( y + k \).
It then follows that \( \bar{x} \to x \) and \( \bar{y} \to y \) as \( (h, k) \to (0, 0) \). Consequently, since the second partial derivatives of \( f \) are continuous,

\[
\lim_{(h,k) \to (0,0)} \frac{S(h, k)}{hk} = \frac{\partial^2 f}{\partial y \partial x}(x, y).
\]

\( \square \)

(d) Deduce that if \( f \) is of class \( C^2 \), then

\[
\frac{\partial^2 f}{\partial y \partial x}(x, y) = \frac{\partial^2 f}{\partial x \partial y}(x, y);
\]

that is, the mixed second partial derivatives are the same for \( C^2 \) maps.

Solution: An argument similar to that in Problem 9(a), with \( g(y) = f(x + h, y) - f(x, y) \), leads to

\[
S(h, k) = \left( \frac{\partial f}{\partial y}(x + h, \bar{y}) - \frac{\partial f}{\partial y}(x, \bar{y}) \right) k,
\]

for some \( \bar{y} \) between \( y \) and \( y + k \). Consequently, by the Mean Value Theorem again, there exists \( \bar{x} \) between \( x \) and \( x + h \) such that

\[
S(h, k) = \frac{\partial^2 f}{\partial x \partial y}(\bar{x}, \bar{y})hk.
\]

Hence, the argument used in the previous part yields that, if the second partial derivatives of \( f \) are continuous,

\[
\lim_{(h,k) \to (0,0)} \frac{S(h, k)}{hk} = \frac{\partial^2 f}{\partial x \partial y}(x, y).
\]

It then follows that, if \( f \) is of class \( C^2 \),

\[
\frac{\partial^2 f}{\partial y \partial x}(x, y) = \frac{\partial^2 f}{\partial x \partial y}(x, y).
\]

\( \square \)