Information Sheet for Exam 2

1. **Jacobian Matrix of a C^1 Function**

   The Jacobian matrix of a function $\Phi\colon D \to \mathbb{R}^2$ defined on an open subset, D, of $\mathbb{R}^2$ by
   $$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in D,$$
   where x and y are C^1 scalar fields on D, is given by
   $$D\Phi(u,v) = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix},$$
   where the partial derivatives are evaluated at (u,v) in D.

2. **Jacobian Determinant of a C^1 Function**

   The Jacobian determinant, or simply the Jacobian, of a C^1 function $\Phi\colon D \to \mathbb{R}^2$ is the determinant of the Jacobian matrix $D\Phi(u,v)$. We denote it by $\dfrac{\partial(x,y)}{\partial(u,v)}$.

3. **Tangent Line Approximation to a C^1 Path**

   The tangent line approximation to a C^1 path $\sigma\colon [a,b] \to \mathbb{R}^n$ at $\sigma(t_o)$, for some $t_o \in (a,b)$, is the straight line given by
   $$L(t) = \sigma(t_o) + (t - t_o)\sigma'(t_o) \quad \text{for all} \quad t \in \mathbb{R}$$

4. **Arc Length**

   Let $\sigma\colon [a,b] \to \mathbb{R}^n$ be a C^1 parametrization of a curve C. The arc length of C is given by
   $$\ell(C) = \int_a^b \|\sigma'(t)\|\ \mathrm{d}t.$$

5. **Path Integral**

   Let $f\colon U \to \mathbb{R}$ be a continuous scalar field defined on some open subset of $\mathbb{R}^n$. Suppose there is a C^1 curve C contained in U. Then the integral of f over C is given by
   $$\int_C f\ \mathrm{d}s = \int_a^b f(\sigma(t))\|\sigma'(t)\|\ \mathrm{d}t,$$
   for any C^1 parametrization, $\sigma\colon [a,b] \to \mathbb{R}^n$ of the curve C.

6. **Line Integral**

   Let $F\colon U \to \mathbb{R}^n$ denote a continuous vector field defined on some open subset, U, of $\mathbb{R}^n$. Suppose there is a C^1 curve, C, contained in U. Then, the line integral of F over C is given by

$$\int_C F\cdot T\ \mathrm{d}s = \int_a^b F(\sigma(t))\cdot\sigma'(t)\ \mathrm{d}t,$$

   for any C^1 parametrization, $\sigma\colon [a,b] \to \mathbb{R}^n$, of the curve C. Here T denotes the tangent unit vector to the curve, and it is given by

$$T(t) = \frac{1}{\|\sigma'(t)\|}\sigma'(t) \quad \text{for all} \ \ t \in (a,b).$$

   If $F = P\,\widehat{\imath} + Q\,\widehat{\jmath} + R\,\widehat{k}$, where P Q and R are C^1 scalar fields defined on U,

$$\int_C F\cdot T\ \mathrm{d}s = \int_C P\ \mathrm{d}x + Q\ \mathrm{d}y + R\ \mathrm{d}z.$$

   The expression $P\ \mathrm{d}x + Q\ \mathrm{d}y + R\ \mathrm{d}z$ is called a differential 1–form.

7. **Flux**

   Let $F = P\,\widehat{\imath} + Q\,\widehat{\jmath}$, where P and Q are continuous scalar fields defined on an open subset, U, of $\mathbb{R}^2$. Suppose there is a C^1 simple closed curve C contained in U. Then the flux of F across C is given by

$$\int_C F\cdot\widehat{n}\ \mathrm{d}s = \int_C P\mathrm{d}y - Q\mathrm{d}x.$$

   Here, $\widehat{n}$ denotes a unit vector perpendicular to C and pointing to the outside of C.

8. **Green's Theorem.**

   The Fundamental Theorem of Calculus,

$$\int_M \mathrm{d}\omega = \int_{\partial M}\omega,$$

   takes the following form in two–dimensional Euclidean space:

   Let R denote a region in $\mathbb{R}^2$ bounded by a simple closed curve, $\partial\mathbb{R}$, made up of a finite number of C^1 paths traversed in the counterclockwise sense. Let P

and Q denote two C^1 scalar fields defined on some open set containing R and its boundary, ∂R. Then,

$$\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y = \int_{\partial R} P\mathrm{d}x + Q\mathrm{d}y.$$

9. **The Change of Variables Theorem**

   Let R denote a region in the xy–plane and D a region in the uv–plane. Suppose that there is a change or coordinates function $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ that maps D onto R. Then, for any continuous function, f, defined on R,

$$\int_R f(x,y)\ \mathrm{d}x\mathrm{d}y = \int_D f(\Phi(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ \mathrm{d}u\mathrm{d}v,$$

   where $\dfrac{\partial(x,y)}{\partial(u,v)}$ denotes the determinant of the Jacobian matrix of Φ.

10. **Polar Coordinates**

    Suppose the change of variable

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

    maps the region D in the $r\theta$–plane onto the region R in the xy–plane in a one–to–one fashion. Then, for any continuous function, f, defined on R,

$$\int_R f(x,y)\ \mathrm{d}x\mathrm{d}y = \int_D f(r\cos\theta, r\sin\theta)\ r\ \mathrm{d}r\mathrm{d}\theta.$$