## Information Sheet for Final Exam

1. **Euclidean Norm and Inner Product**

   Let $v$ and $w$ denote vectors in $\mathbb{R}^n$, $v \cdot w$ denote the dot product of $v$ and $w$, and $\|v\|$ and $\|w\|$ their respective norms; then,

   (a) **Cauchy–Schwarz Inequality**

   $$|v \cdot w| \leqslant \|v\|\|w\|$$

   (b) **Triangle Inequality**

   $$\|v + w\| \leqslant \|v\| + \|w\|$$

2. **Cross–Product in $\mathbb{R}^3$**

   For vectors $v$ and $w$ in $\mathbb{R}^3$, the cross–product, $v \times w \in \mathbb{R}^3$, of $v$ and $w$ is antisymmetric (i.e., $w \times v = -v \times w$), bilinear, and satisfies:

   $$\widehat{i} \times \widehat{j} = \widehat{k}, \quad \widehat{k} \times \widehat{i} = \widehat{j}, \quad \text{and} \quad \widehat{j} \times \widehat{k} = \widehat{i}.$$

3. **Planes in $\mathbb{R}^3$**

   The equation of a plane through $P_o(x_o, y_o, z_o)$ and perpendicular to the vector $n = a\,\widehat{i} + b\,\widehat{j} + c\,\widehat{k}$ is given by

   $$a(x - x_o) + b(y - y_o) + c(z - z_o) = 0.$$

4. **Jacobian Matrix of a $C^1$ Function**

   The Jacobian matrix of a function $\Phi\colon D \to \mathbb{R}^2$ defined on an open subset, $D$, of $\mathbb{R}^2$ by

   $$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in D,$$

   where $x$ and $y$ are $C^1$ scalar fields on $D$, is given by

   $$D\Phi(u,v) = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix},$$

   where the partial derivatives are evaluated at $(u, v)$ in $D$.

5. **Jacobian Determinant of a $C^1$ Function**

   The Jacobian determinant, or simply the Jacobian, of a $C^1$ function $\Phi\colon D \to \mathbb{R}^2$ is the determinant of the Jacobian matrix $D\Phi(u, v)$. We denote it by $\dfrac{\partial(x, y)}{\partial(u, v)}$.

6. **Tangent Line Approximation to a $C^1$ Path**

   The tangent line approximation to a $C^1$ path $\sigma\colon [a, b] \to \mathbb{R}^n$ at $\sigma(t_o)$, for some $t_o \in (a, b)$, is the straight line given by

   $$L(t) = \sigma(t_o) + (t - t_o)\sigma'(t_o) \quad \text{for all} \ \ t \in \mathbb{R}$$

7. **Arc Length**

   Let $\sigma\colon [a, b] \to \mathbb{R}^n$ be a $C^1$ parametrization of a curve $C$. The arc length of $C$ is given by

   $$\ell(C) = \int_a^b \|\sigma'(t)\| \ \mathrm{d}t.$$

8. **Path Integral**

   Let $f\colon U \to \mathbb{R}$ be a continuous scalar field defined on some open subset of $\mathbb{R}^n$. Suppose there is a $C^1$ curve $C$ contained in $U$. Then the integral of $f$ over $C$ is given by

   $$\int_C f \ \mathrm{d}s = \int_a^b f(\sigma(t))\|\sigma'(t)\| \ \mathrm{d}t,$$

   for any $C^1$ parametrization, $\sigma\colon [a, b] \to \mathbb{R}^n$ of the curve $C$.

9. **Line Integral**

   Let $F\colon U \to \mathbb{R}^n$ denote a continuous vector field defined on some open subset, $U$, of $\mathbb{R}^n$. Suppose there is a $C^1$ curve, $C$, contained in $U$. Then, the line integral of $F$ over $C$ is given by

   $$\int_C F \cdot T \ \mathrm{d}s = \int_a^b F(\sigma(t)) \cdot \sigma'(t) \ \mathrm{d}t,$$

   for any $C^1$ parametrization, $\sigma\colon [a, b] \to \mathbb{R}^n$, of the curve $C$. Here $T$ denotes the tangent unit vector to the curve, and it is given by

   $$T(t) = \frac{1}{\|\sigma'(t)\|}\sigma'(t) \quad \text{for all} \ \ t \in (a, b).$$

If $F = P\,\widehat{i} + Q\,\widehat{j} + R\,\widehat{k}$, where $P$ $Q$ and $R$ are $C^1$ scalar fields defined on $U$,

$$\int_C F\cdot T\ \mathrm{d}s = \int_C P\ \mathrm{d}x + Q\ \mathrm{d}y + R\ \mathrm{d}z.$$

The expression $P\ \mathrm{d}x + Q\ \mathrm{d}y + R\ \mathrm{d}z$ is called a differential 1–form.

10. **Flux**

Let $F = P\,\widehat{i} + Q\,\widehat{j}$, where $P$ and $Q$ are continuous scalar fields defined on an open subset, $U$, of $\mathbb{R}^2$. Suppose there is a $C^1$ simple closed curve $C$ contained in $U$. Then the flux of $F$ across $C$ is given by

$$\int_C F\cdot\widehat{n}\ \mathrm{d}s = \int_C P\mathrm{d}y - Q\mathrm{d}x.$$

Here, $\widehat{n}$ denotes a unit vector perpendicular to $C$ and pointing to the outside of $C$.

11. **Green's Theorem.**

The Fundamental Theorem of Calculus,

$$\int_M \mathrm{d}\omega = \int_{\partial M}\omega,$$

takes the following form in two–dimensional Euclidean space:

Let $R$ denote a region in $\mathbb{R}^2$ bounded by a simple closed curve, $\partial\mathbb{R}$, made up of a finite number of $C^1$ paths traversed in the counterclockwise sense. Let $P$ and $Q$ denote two $C^1$ scalar fields defined on some open set containing $R$ and its boundary, $\partial R$. Then,

$$\int_R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \int_{\partial R} P\mathrm{d}x + Q\mathrm{d}y.$$

12. **The Divergence Theorem in $\mathbb{R}^2$.**

Let $R$ denote a region in $\mathbb{R}^2$ bounded by a simple closed curve, $\partial\mathbb{R}$, made up of a finite number of $C^1$ paths traversed in the counterclockwise sense. The flux of a vector field $F = P\widehat{i} + Q\widehat{j}$ across $\partial R$, where $P$ and $Q$ are $C^1$ scalar fields defined on some open set containing $R$ and its boundary, $\partial R$, is defined by

$$\oint_{\partial R} F\cdot\widehat{n}\ \mathrm{d}s = \oint_{\partial R} P\ \mathrm{d}y - Q\ \mathrm{d}x.$$

The Theorem of Calculus in this case takes the form

$$\oint_{\partial R} F \cdot \widehat{n} \, \mathrm{d}s = \int_R \mathrm{div} F \, \mathrm{d}x\mathrm{d}y,$$

where $\mathrm{div}F = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ is the divergence of the $C^1$ field $F$.

13. **The Change of Variables Theorem**

    Let $R$ denote a region in the $xy$–plane and $D$ a region in the $uv$–plane. Suppose that there is a change or coordinates function $\Phi \colon \mathbb{R}^2 \to \mathbb{R}^2$ that maps $D$ onto $R$. Then, for any continuous function, $f$, defined on $R$,

    $$\int_R f(x,y) \, \mathrm{d}x\mathrm{d}y = \int_D f(\Phi(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, \mathrm{d}u\mathrm{d}v,$$

    where $\dfrac{\partial(x,y)}{\partial(u,v)}$ denotes the determinant of the Jacobian matrix of $\Phi$.

14. **Polar Coordinates**

    Suppose the change of variable

    $$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

    maps the region $D$ in the $r\theta$–plane onto the region $R$ in the $xy$–plane in a one–to–one fashion. Then, for any continuous function, $f$, defined on $R$,

    $$\int_R f(x,y) \, \mathrm{d}x\mathrm{d}y = \int_D f(r\cos\theta, r\sin\theta) \; r \; \mathrm{d}r\mathrm{d}\theta.$$