Solutions to Assignment #10

1. Let $U = \mathbb{R}^n \setminus \{0\} = \{v \in \mathbb{R}^n \mid v \neq 0\}$ and define $f: \mathbb{R}^n \to \mathbb{R}$ by

$$f(v) = \|v\| \text{ for all } v \in \mathbb{R}.$$ 

(a) Prove that $f$ is differentiable on $U$.

**Solution:** For $v = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$, write

$$f(v) = f(x_1, x_2, \ldots, x_n) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

and observe that if $(x_1, x_2, \ldots, x_n) \in U$, then $x_1^2 + x_2^2 + \cdots + x_n^2 \neq 0$ so that the partial derivatives

$$\frac{\partial f}{\partial x_j}(x_1, x_2, \ldots, x_n) = \frac{x_j}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}, \quad j = 1, 2, \ldots, n,$$

exist in $U$ and are continuous there. Therefore, $f$ is a $C^1$ map in $U$ and it is therefore differentiable in $U$. □

(b) Prove that $f$ is not differentiable at the origin in $\mathbb{R}^n$.

**Solution:** Arguing by contradiction, assume that $f$ is differentiable at the origin. Then, there exists a linear transformation, $T: \mathbb{R}^n \to \mathbb{R}$ such that

$$f(w) = T(w) + E_o(w), \quad (1)$$

for $\|w\|$ small, where

$$\lim_{\|w\| \to 0} \frac{\|E_o(w)\|}{\|w\|} = 0. \quad (2)$$

Take $w = te_j$, where $e_j$ is one of the standard basis vectors. It then follows from (1) that

$$|t| = tT(e_j) + E_o(te_j),$$

for $t \in \mathbb{R}$ with $|t|$ sufficiently small. Thus, if $t \neq 0$ and $|t|$ is sufficiently small,

$$\frac{|t|}{t} = T(e_j) + \frac{1}{t}E_o(te_j).$$
Observe that, by (2),

$$\lim_{t \to 0} \frac{1}{t} E_{\alpha}(te_j) = 0.$$  

Hence,

$$\lim_{t \to 0} \frac{|t|}{t} = T(e_j),$$

which is impossible since $\lim_{t \to 0} \frac{|t|}{t}$ does not exist. Consequently, $f(v) = \|v\|$ is not differentiable at the origin. \hfill \Box

2. Let $I$ be an open interval of real numbers, and suppose that $\sigma: I \to \mathbb{R}^n$ is a differentiable path satisfying $\sigma(t) \neq \mathbf{0}$ for all $t \in I$. Show that the function $g: I \to \mathbb{R}$ defined by $g(t) = \|\sigma(t)\|$ for all $t \in I$ is differentiable on $I$ and compute its derivative.

**Solution:** Let $f$ be as defined in Problem (1a) and observe that

$$g = f \circ \sigma.$$  

Note that, since $\sigma(t) \neq \mathbf{0}$ for all $t \in I$, $\sigma(I) \subseteq U$, where $U$ is as defined in Problem 1a. Consequently, by the result of part (a) in Problem 1a, $g$ is differentiable on $I$, by the Chain Rule, and its derivative is given by

$$Dg(t) = Df(\sigma(t))D\sigma(t) = \nabla f(\sigma(t)) \cdot \sigma'(t), \quad \text{for all } t \in I,$$

where

$$\nabla f(v) = \frac{1}{\|v\|} v \quad \text{for all } v \in U.$$  

We then have that

$$g'(t) = \frac{1}{\|\sigma(t)\|} \sigma(t) \cdot \sigma'(t), \quad \text{for all } t \in I.$$  

\hfill \Box
3. Let $I$ be an open interval of real numbers and $U$ be an open subset of $\mathbb{R}^n$. Suppose that $\sigma: I \rightarrow \mathbb{R}^n$ is a differentiable path and that $f: U \rightarrow \mathbb{R}$ is a differentiable scalar field. Assume also that the image of $I$ under $\sigma$, $\sigma(I)$, is contained in $U$. Suppose also that the derivative of the path $\sigma$ satisfies 
\[ \sigma'(t) = -\nabla f(\sigma(t)) \quad \text{for all} \quad t \in I. \]
Show that if the gradient of $f$ along the path $\sigma$ is never zero, then $f$ decreases along the path as $t$ increases.

*Suggestion:* Use the Chain Rule to compute the derivative of $f(\sigma(t))$.

**Solution:** Using the Chain Rule to compute the derivative of $f(\sigma(t))$
we obtain that
\[ \frac{d}{dt}(f(\sigma(t))) = \nabla f(\sigma(t)) \cdot \sigma'(t) = -\nabla f(\sigma(t)) \cdot \nabla f(\sigma(t)) = -\|\nabla f(\sigma(t))\|^2. \]

It then follows that
\[ \frac{d}{dt}(f(\sigma(t))) < 0 \quad \text{for all} \quad t \in I, \]
and therefore $f(\sigma(t))$ decreases as $t$ increases. \(\square\)

4. A set $U \subseteq \mathbb{R}^n$ is said to be **path connected** iff for any vectors $x$ and $y$ in $U$, there exists a differentiable path $\sigma: [0, 1] \rightarrow \mathbb{R}^n$ such that $\sigma(0) = x$, $\sigma(1) = y$ and $\sigma(t) \in U$ for all $t \in [0, 1]$; i.e., any two elements in $U$ can be connected by a differentiable path whose image is entirely contained in $U$.

Suppose that $U$ is an open, path connected subset of $\mathbb{R}^n$. Let $f: U \rightarrow \mathbb{R}$ be a differentiable scalar field such that $\nabla f(x)$ is the zero vector for all $x \in U$. Prove that $f$ must be constant.

**Solution:** Fix a point $x_o$ in $U$. Then, for any $x \in U$, there exists a differentiable path $\sigma: [0, 1] \rightarrow \mathbb{R}^n$ such that $\sigma(0) = x_o$, $\sigma(1) = x$, and $\sigma(t) \in U$ for all $t \in [0, 1]$. Then $f(\sigma(t))$ for all $t \in I$ defines a differentiable function on $(0, 1)$ with
\[ \frac{d}{dt}(f(\sigma(t))) = \nabla f(\sigma(t)) \cdot \sigma'(t) \quad \text{for all} \quad t \in (0, 1). \]
Thus, since $\nabla f(y) = 0$ for all $y \in U$, it follows that

$$\frac{d}{dt}(f(\sigma(t))) = 0 \quad \text{for all } t \in (0,1),$$

which implies that $f(\sigma(t))$ is constant on $(0,1)$. It then follows, by continuity, that

$$f(\sigma(1)) = f(\sigma(0)),$$

or

$$f(x) = f(x_0).$$

Since this is true for every $x \in U$, we conclude that $f$ is constant on $U$. \qed

5. (Exercises 2 and 4 on page 207 in the text).

(a) (Exercise 2 on pg. 207) Let $x$ and $y$ be functions of $u$ and $v$: $x = x(u,v)$, $y = y(u,v)$, and let $f(x,y)$ be a scalar field. Find $\partial f/\partial u$ and $\partial f/\partial v$ in terms of $\partial f/\partial x$, $\partial f/\partial y$, $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, and $\partial y/\partial v$.

**Solution**: Apply the Chain Rule to $f(x(u,v),y(u,v))$ to get that

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}$$

and

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.$$

\qed

(b) (Exercise 4 on pg. 207) For $f$ $x$ and $y$ as in Exercise 2, express $\partial^2 f/\partial u^2$ in terms of the partial derivatives of $f$ with respect to $x$ and $y$ and the partial derivatives of $x$ and $y$ with respect to $u$. Assume that

$$\partial^2 f/\partial x \partial y = \partial^2 f/\partial y \partial x.$$
Solution: Apply the Product Rule and the Chain Rule to get

\[
\frac{\partial^2 f}{\partial u^2} = \frac{\partial}{\partial u} \left( \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} \right)
\]

\[
= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} \frac{\partial}{\partial u} \left( \frac{\partial x}{\partial u} \right) + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} \frac{\partial}{\partial u} \left( \frac{\partial y}{\partial u} \right)
\]

\[
+ \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} \frac{\partial}{\partial u} \left( \frac{\partial x}{\partial u} \right) + \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} \frac{\partial}{\partial u} \left( \frac{\partial y}{\partial u} \right)
\]

\[
= \frac{\partial f}{\partial x} \frac{\partial^2 x}{\partial u^2} \left( \frac{\partial^2 f}{\partial^2 x} \frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial x \partial y} \frac{\partial y}{\partial u} \right) \frac{\partial x}{\partial u}
\]

\[
+ \frac{\partial f}{\partial y} \frac{\partial^2 y}{\partial u^2} \left( \frac{\partial^2 f}{\partial^2 y} \frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial y \partial x} \frac{\partial x}{\partial u} \right) \frac{\partial y}{\partial u}
\]

\[
= \frac{\partial f}{\partial x} \frac{\partial^2 x}{\partial u^2} \left( \frac{\partial x}{\partial u} \right)^2 + 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial u} \frac{\partial y}{\partial u}
\]

\[
+ \frac{\partial f}{\partial y} \frac{\partial^2 y}{\partial u^2} \left( \frac{\partial y}{\partial u} \right)^2 .
\]