Assignment #11

Due on Wednesday, November 11, 2009

Read Section 3.1 on *The Calculus of Curves*, pp. 53–65, in Bressoud.
Read Section 5.2 on *Line Integrals*, pp. 113–119, in Bressoud.

Do the following problems

1. Let $I$ denote an open interval in $\mathbb{R}$, and $\sigma: I \to \mathbb{R}^n$ be a $C^1$ path. For fixed $a \in I$, define

$$s(t) = \int_a^t \|\sigma'(\tau)\| \, d\tau \quad \text{for all} \quad t \in I.$$ 

Show that $s$ is differentiable and compute $s'(t)$ for all $t \in I$.

2. Let $\sigma$ and $s$ be as defined in the previous problem. Suppose, in addition, that $\sigma'(t)$ is never the zero vector for all $t$ in $I$. Show that $s$ is a strictly increasing function of $t$ and that it is, therefore, one–to–one.

3. Let $\sigma$ and $s$ be as defined in Problem 1. We can re–parameterize $\sigma$ by using $s$ as a parameter. We therefore obtain $\sigma(s)$, where $s$ is the *arc length* parameter.

Differentiate the expression

$$\sigma(s(t)) = \sigma(t)$$

with respect to $t$ using the Chain Rule. Conclude that, if $\sigma'(t)$ is never the zero vector for all $t$ in $I$, then $\sigma'(s)$ is always a unit vector.

The vector $\sigma'(s)$ is called the *unit tangent vector* to the path $\sigma$.

4. For $a$ and $b$, positive real numbers, the expression

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

defines an ellipse in the $xy$–plane $\mathbb{R}^2$.

Sketch the ellipse, give a parametrization for it, and set up the integral that yields its arc length.

5. Let $\sigma: [0, \pi] \to \mathbb{R}^3$ be defined by $\sigma(t) = t \hat{i} + t \sin t \hat{j} + t \cos t \hat{k}$ for all $t \in [0, \pi]$.

Compute the arc length of the curve parametrized by $\sigma$. 