1. Let $I$ denote an open interval in $\mathbb{R}$, and $\sigma: I \to \mathbb{R}^n$ be a $C^1$ path. For fixed $a \in I$, define

$$s(t) = \int_a^t \|\sigma'(\tau)\| \, d\tau \quad \text{for all } t \in I.$$ 

Show that $s$ is differentiable and compute $s'(t)$ for all $t \in I$.

**Solution:** Since the $\sigma$ is a $C^1$ path, the map $t \mapsto \|\sigma'(t)\|$ is continuous on $I$. Therefore, by the Fundamental Theorem of Calculus, $s(t)$ is differentiable and

$$s'(t) = \frac{d}{dt} \int_a^t \|\sigma'(\tau)\| \, d\tau = \|\sigma'(t)\| \quad \text{for all } t \in I.$$

2. Let $\sigma$ and $s$ be as defined in the previous problem. Suppose, in addition, that $\sigma'(t)$ is never the zero vector for all $t$ in $I$. Show that $s$ is a strictly increasing function of $t$ and that it is, therefore, one–to–one.

**Solution:** From the previous problem,

$$s'(t) = \|\sigma'(t)\| \quad \text{for all } t \in I,$$

so that, since $\sigma'(t)$ is never the zero vector for all $t$ in $I$, $s'(t) > 0$ for all $t \in I$. It then follows that $s$ is a strictly increasing function of $t$ and, therefore, it is a one–to–one map.

3. Let $\sigma$ and $s$ be as defined in Problem 1. We can re–parameterize $\sigma$ by using $s$ as a parameter. We therefore obtain $\sigma(s)$, where $s$ is the *arc length* parameter. Differentiate the expression

$$\sigma(s(t)) = \sigma(t)$$

with respect to $t$ using the Chain Rule. Conclude that, if $\sigma'(t)$ is never the zero vector for all $t$ in $I$, then $\sigma'(s)$ is always a unit vector.

The vector $\sigma'(s)$ is called the *unit tangent vector* to the path $\sigma$. 
Solution: Differentiate

\[ \sigma(s(t)) = \sigma(t) \]

with respect to \( t \) to get

\[ \frac{d}{dt} \sigma(s(t)) = \sigma'(t); \]

thus, by the Chain Rule,

\[ \sigma'(s) s'(t) = \sigma'(t), \]

or

\[ \sigma'(s) \|\sigma'(t)\| = \sigma'(t). \]

Since \( \|\sigma'(t)\| \neq 0 \) for all \( t \in I \), we have that

\[ \sigma'(s) = \frac{1}{\|\sigma'(t)\|} \sigma'(t), \]

and therefore \( \sigma'(s) \) is a unit vector. \( \square \)

4. For \( a \) and \( b \), positive real numbers, the expression

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

defines an ellipse in the \( xy \)-plane \( \mathbb{R}^2 \).

Sketch the ellipse, give a parametrization for it, and set up the integral that yields its arc length.

Solution: A sketch of the ellipse for the case \( b < a \) is shown in Figure 1.

The path

\[ \sigma(t) = (a \cos t, b \sin t) \quad \text{for all } \ t \in [0, 2\pi] \]

is a \( C^1 \) parametrization of the ellipse. The arc length of the ellipse is then given by

\[ \int_0^{2\pi} \|\sigma'(t)\| \, dt, \]

where

\[ \sigma'(t) = (-a \sin t, b \cos t) \quad \text{for all } \ t \in \mathbb{R}, \]
Thus
\[
\int_0^{2\pi} \|\sigma'(t)\| \, dt = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \, dt.
\]
\[\Box\]

5. Let \( \sigma: [0, \pi] \to \mathbb{R}^3 \) be defined by \( \sigma(t) = t \hat{i} + t \sin t \hat{j} + t \cos t \hat{k} \) for all \( t \in [0, \pi] \). Compute the arc length of the curve parametrized by \( \sigma \).

**Solution:** Let \( C \) denote the curve parametrized by \( \sigma \); then,
\[
\ell(C) = \int_0^\pi \|\sigma'(t)\| \, dt,
\]
where
\[
\sigma'(t) = \hat{i} + (\sin t + t \cos t) \hat{j} + (\cos t - t \sin t) \hat{k}
\]
for all \( t \in \mathbb{R} \), and therefore
\[
\|\sigma'(t)\| = \sqrt{1 + (\sin t + t \cos t)^2 + (\cos t - t \sin t)^2}
\]
\[
= \sqrt{2 + t^2}.
\]
Thus,
\[
\ell(C) = \int_0^\pi \sqrt{2 + t^2} \, dt
\]
\[
= \left[ \frac{t}{2} \sqrt{2 + t^2} + \ln |t + \sqrt{2 + t^2}| \right]_0^\pi
\]
\[
= \frac{\pi}{2} \sqrt{2 + \pi^2} + \ln(\pi + \sqrt{2 + \pi^2}) - \frac{1}{2} \ln 2.
\]