Solutions to Assignment #14

1. (Exercise 4 on page 119 in the text)

Integrate the 1–form \( yz \, dx + xz \, dy + xy \, dz \) over each of the following paths from \((0,1,0)\) to \((2,1,1)\).

(a) the straight line from \((0,1,0)\) to \((2,1,1)\),
(b) the lines from \((0,1,0)\) to \((0,1,1)\) to \((2,1,1)\),
(c) the lines from \((0,1,0)\) to \((2,1,0)\) to \((2,1,1)\),
(d) the arc \((2t, (2t - 1)^2, t)\), for \(0 \leq t \leq 1\).

Solution: The integral \( \int_C yz \, dx + xz \, dy + xy \, dz \) is the line integral of the gradient field \( \nabla f \), where \( f(x,y,z) = xyz \) for all \((x,y,z) \in \mathbb{R}^3\). It then follows that, for any curve, \( C \), connecting \((0,1,0)\) to \((2,1,1)\),

\[
\int_C yz \, dx + xz \, dy + xy \, dz = f(2,1,1) - f(0,1,0) = 2.
\]

So, the answer all four parts must be 2.

2. (Exercises 6(d)(e)(f) on pages 119 and 120 in the text)

Integrate the 1–form

\[
-\frac{y \, dx + x \, dy}{x^2 + y^2}
\]

over each of the following paths from \((-1,0)\) to \((1,0)\):

(d) the lines from \((-1,0)\) to \((0,-1)\) to \((1,0)\),

Solution: The curve \( C \) connecting \((-1,0)\) to \((1,0)\) given here is made up of two lines:

(i) \( C_1 \): the line segment from \((-1,0)\) to \((0,-1)\) and parametrized by

\[
\begin{align*}
    x &= t - 1 \\
    y &= -t
\end{align*}
\]

for \(0 \leq t \leq 1\). Then,

\[
\begin{align*}
    dx &= \, dt \\
    dy &= - \, dt
\end{align*}
\]
and
\[
\int_{C_1} \frac{-y \, dx + x \, dy}{x^2 + y^2} = \int_0^1 \frac{1}{(t-1)^2 + t^2} \, dt
= \int_0^1 \frac{1}{2t^2 - 2t + 1} \, dt
= \int_0^1 \frac{1}{2(t-1/2)^2 + 1/2} \, dt
= 2 \int_0^1 \frac{1}{(2t-1)^2 + 1} \, dt.
\]

Next, make the change of variables \( u = 2t - 1 \) to get that
\[
\int_{C_1} \frac{-y \, dx + x \, dy}{x^2 + y^2} = \int_{-1}^{1} \frac{1}{u^2 + 1} \, du
= [\arctan(u)] + (-1)^1
= \arctan(1) - \arctan(-1)
= \frac{\pi}{4} - \left( -\frac{\pi}{4} \right)
= \frac{\pi}{2}.
\]

(ii) \( C_2 \): the line segment from \((0, -1)\) to \((1, 0)\) and parametrized by
\[
\begin{cases} 
  x = t \\
  y = t - 1 
\end{cases}
\]
for \(0 \leq t \leq 1\). Then,
\[
\begin{cases} 
  dx = dt \\
  dy = dt 
\end{cases}
\]
and
\[
\int_{C_2} \frac{-y \, dx + x \, dy}{x^2 + y^2} = \int_0^1 \frac{1}{(t-1)^2 + t^2} \, dt = \frac{\pi}{2}.
\]
by the same calculations used in the previous part. It then follows that
\[ \int_C \frac{-y \, dx + x \, dy}{x^2 + y^2} = \int_{C_1} \frac{-y \, dx + x \, dy}{x^2 + y^2} + \int_{C_2} \frac{-y \, dx + x \, dy}{x^2 + y^2} = \pi. \]
\[ \square \]

(e) the curve \((-\cos t, \sin t), 0 \leq t \leq \pi,\)

**Solution:** Let \(C\) denote the curve parametrized by
\[
\begin{cases}
x = -\cos t \\
y = \sin t
\end{cases}
\]
for \(0 \leq t \leq \pi\). Then,
\[
\begin{cases}
dx = \sin t \, dt \\
dy = \cos t \, dt
\end{cases}
\]
and
\[
\int_C \frac{-y \, dx + x \, dy}{x^2 + y^2} = \int_0^\pi \frac{-\sin^2 t - \cos^2 t}{\cos^2 t + \sin^2 t} \, dt = -\pi.
\]
\[ \square \]

(f) the curve \((-\cos t, -\sin t), 0 \leq t \leq \pi,\)

**Solution:** Let \(C\) denote the curve parametrized by
\[
\begin{cases}
x = -\cos t \\
y = -\sin t
\end{cases}
\]
for \(0 \leq t \leq 1\). Then,
\[
\begin{cases}
dx = \sin t \, dt \\
dy = -\cos t \, dt
\end{cases}
\]
and
\[
\int_C \frac{-y \, dx + x \, dy}{x^2 + y^2} = \int_0^\pi \frac{\sin^2 t + \cos^2 t}{\cos^2 t + \sin^2 t} \, dt = \pi.
\]
\[ \square \]
3. Let $\sigma : [a, b] \rightarrow \mathbb{R}^n$ be a $C^1$ parametrization of a curve $C$ in $\mathbb{R}^n$. Let $h : [c, d] \rightarrow [a, b]$ be a one–to–one and onto map such that $h'(t) > 0$ for all $t \in [c, d]$. Define 

$$\gamma(t) = \sigma(h(t)) \quad \text{for all } t \in [c, d].$$

$\gamma : [c, d] \rightarrow \mathbb{R}^n$ is called a reparametrization of $\sigma$.

Let $F : U \rightarrow \mathbb{R}^n$ denote a continuous vector field defined on a region $U$ of $\mathbb{R}^n$ which contains the curve $C$. Show that

$$\int_a^b F(\sigma(t)) \cdot \sigma'(t) \, dt = \int_c^d F(\gamma(t)) \cdot \gamma'(t) \, dt.$$

Thus, the line integral

$$\int_C F \cdot T \, ds$$

is independent of reparametrization.

**Solution:** Use the Chain Rule and the fact that $\gamma(t) = \sigma(h(t))$ for all $t \in [a, b]$ to get that

$$\int_a^b F(\gamma(t)) \cdot \gamma'(t) \, dt = \int_a^b F(\sigma(h(t))) \cdot \sigma'(h(t)) \, h'(t) \, dt.$$

Next, make the change of variables $\tau = h(t)$, for all $t \in [a, b]$, to get that $d\tau = h'(t) \, dt$ and

$$\int_a^b F(\gamma(t)) \cdot \gamma'(t) \, dt = \int_{h(a)}^{h(b)} F(\sigma(\tau)) \cdot \sigma'(\tau) \, d\tau.$$

Now, since $h'(t) > 0$ for all $t$, then $h$ is increasing on $[a, b]$ and, therefore, $h(a) = c$ and $h(b) = d$. Consequently,

$$\int_a^b F(\gamma(t)) \cdot \gamma'(t) \, dt = \int_c^d F(\sigma(\tau)) \cdot \sigma'(\tau) \, d\tau.$$

□

4. Let $\sigma : [0, 1] \rightarrow \mathbb{R}^n$ be a $C^1$ parametrization of a curve $C$ is $\mathbb{R}^n$. Give a $C^1$ reparametrization, $\gamma : [0, 1] \rightarrow \mathbb{R}^n$, of $\sigma$ in which the curve $C$ is traversed in the opposite direction as that of $\sigma$. What is $\gamma'$ in terms of $\sigma''$?
**Solution:** Let $\gamma : [0, 1] \to \mathbb{R}^n$ be given by

$$\gamma(t) = \sigma(\tau), \quad \text{where} \quad \tau = 1 - t, \quad \text{for} \quad t \in [0, 1].$$

Then, by the Chain Rule,

$$\gamma'(t) = -\sigma'(\tau) \quad \text{for} \quad \tau \in [0, 1].$$

Observe that $\gamma(0) = \sigma(1)$ and $\gamma(1) = \sigma(0)$, and so the curve $C$ is traversed in the opposite direction as that of $\sigma$.  \hfill \Box

5. The flux of a 2-dimensional vector field,

$$F(x, y) = P(x, y) \hat{i} + Q(x, y) \hat{j},$$

across a simple, $C^1$, closed curve, $C$, is given by

$$\int_C P \, dy - Q \, dx.$$

Compute the flux of the following fields across the given curves

(a) $F(x, y) = x^2 \hat{i} + y^2 \hat{j}$ and $C$ is the boundary of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$.

**Solution:** We decompose the boundary of the square, $C$, into its edges:

(i) $C_1$: $x = t$, $y = 0$, $0 \leq t \leq 1$. Then

$$\begin{aligned}
\begin{cases}
dx = dt \\
dy = 0 \, dt
\end{cases}
\end{aligned}$$

and

$$\int_{C_1} P \, dy - Q \, dx = \int_{C_1} x^2 \, dy - y^2 \, dx = 0.$$  

(ii) $C_2$: $x = 1$, $y = t$, $0 \leq t \leq 1$. Then

$$\begin{aligned}
\begin{cases}
dx = 0 \, dt \\
dy = dt
\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\int_{C_2} P \, dy - Q \, dx &= \int_{C_2} x^2 \, dy - y^2 \, dx \\
&= \int_0^1 dt = 1.
\end{aligned}$$
(iii) $C_3$: $x = 1 - t, y = 1, 0 \leq t \leq 1$. Then
\[
\begin{aligned}
&\begin{cases}
  dx = -dt \\
  dy = 0 \ dt
\end{cases}
\end{aligned}
\]
and
\[
\int_{C_3} P \ dy - Q \ dx = \int_{C_3} x^2 \ dy - y^2 \ dx
\]
\[
= \int_0^1 dt = 1.
\]

(iv) $C_4$: $x = 0, y = 1 - t, 0 \leq t \leq 1$. Then
\[
\begin{aligned}
&\begin{cases}
  dx = 0 \ dt \\
  dy = -dt
\end{cases}
\end{aligned}
\]
and
\[
\int_{C_4} P \ dy - Q \ dx = \int_{C_4} x^2 \ dy - y^2 \ dx = 0.
\]
It then follows that
\[
\int_C x^2 \ dy - y^2 \ dx = 2.
\]
\[\square\]

(b) $F(x, y) = x \hat{i} + y \hat{j}$ and $C$ is the boundary of the unit circle.

**Solution:** Parametrize $C$ by
\[
\begin{aligned}
&\begin{cases}
  x = \cos t \\
  y = \sin t
\end{cases}
\end{aligned}
\]
for $0 \leq t \leq 2\pi$. Then, \[
\begin{aligned}
&\begin{cases}
  dx = -\sin t \ dt \\
  dy = \cos t \ dt
\end{cases}
\end{aligned}
\]
and
\[
\begin{aligned}
\int_C P \ dy - Q \ dx &= \int_C x \ dy - y \ dx \\
&= \int_0^{2\pi} (\cos^2 t + \sin^2 t) \ dt \\
&= 2\pi.
\end{aligned}
\]
\[\square\]