Assignment #15
Due on Monday, November 30, 2009

Read Section 5.4 on Multiple Integrals, pp. 120–134, in Bressoud.

Background and Definitions

- **Flux**
  Let \( F = P \hat{i} + Q \hat{j} \), where \( P \) and \( Q \) are continuous scalar fields defined on an open subset, \( U \), of \( \mathbb{R}^2 \). Suppose there is a \( C^1 \) simple closed curve \( C \) contained in \( U \). Then the flux of \( F \) across \( C \) is given by
  \[
  \int_C F \cdot \hat{n} \, ds = \int_C P \, dy - Q \, dx.
  \]
  Here, \( \hat{n} \) denotes a unit vector perpendicular to \( C \) and pointing to the outside of \( C \).

- **Divergence of a Vector Field in \( \mathbb{R}^2 \).**
  Given a \( C^1 \) vector field, \( F(x, y) = P(x, y)\hat{i} + Q(x, y)\hat{j} \), defined on some open subset \( U \) of \( \mathbb{R}^2 \), the divergence of \( F \) is defined to be
  \[
  \text{div} F(x, y) = \frac{\partial P}{\partial x}(x, y) + \frac{\partial Q}{\partial y}(x, y) \quad \text{for all } (x, y) \in U.
  \]

- **Green’s Theorem.**
  Let \( R \) denote a region in \( \mathbb{R}^2 \) bounded by a simple closed curve, \( \partial R \), made up of a finite number of \( C^1 \) paths traversed in the counterclockwise sense. Let \( P \) and \( Q \) denote two \( C^1 \) scalar fields defined on some open set containing \( R \) and its boundary, \( \partial R \). Then,
  \[
  \iint_R \text{div} F \, dx \, dy = \oint_{\partial R} F \cdot \hat{n} \, ds.
  \]

Do the following problems

1. Let \( C \) denote the unit circle traversed in the counterclockwise direction. Evaluate the line integral \( \int_C x^3 \, dy - y^3 \, dx \).
2. Let $F(x, y) = y \mathbf{i} - x \mathbf{j}$ and $R$ be the square in the $xy$–plane with vertices $(0, 0)$, $(2, -1)$, $(3, 1)$ and $(1, 2)$. Evaluate $\int_{\partial R} F \cdot n \, ds$.

3. Consider the iterated integral

$$\int_0^1 \int_y^1 e^{-x^2} \, dx \, dy.$$

(a) Identify the region of integration, $R$, for this integral and sketch it.

(b) Change the order of integration in the iterated integral and evaluate the double integral

$$\int_R e^{-x^2} \, dx \, dy.$$ 

4. What is the region $R$ over which you integrate when evaluating the double integral

$$\int_0^1 \int_{x^2}^1 x \sqrt{1 - y^2} \, dy \, dx?$$

Rewrite this as an iterated integral first with respect to $x$, then with respect to $y$. Evaluate this integral. Which order of integration is easier?

5. What is the region $R$ over which you integrate when evaluating the double integral

$$\int_1^2 \int_1^x \frac{x}{\sqrt{x^2 + y^2}} \, dy \, dx?$$

Rewrite this as an iterated integral first with respect to $x$, then with respect to $y$. Evaluate this integral. Which order of integration is easier?