Solutions to Assignment #15

1. Let $C$ denote the unit circle traversed in the counterclockwise direction. Evaluate the line integral $\int_C x^3 \, dy - y^3 \, dx$.

**Solution:** Parametrize $C$ by $x = \cos t$, $y = \sin t$ for $0 \leq t \leq 2\pi$. Then,

$$dx = -\sin t \, dt \quad \text{and} \quad dy = \cos t \, dt,$$

and

$$\int_C x^3 \, dy - y^3 \, dx = \int_0^{2\pi} (\cos^4 t + \sin^4 t) \, dt.$$

To evaluate the integral on the right-hand side, first write

$$\cos^4 t = \left[\cos^2 t\right]^2 = \left[\frac{1}{2}(1 + \cos 2t)\right]^2 = \frac{1}{4}(1 + 2 \cos 2t + \cos^2 2t).$$

Similarly,

$$\sin^4 t = \left[\sin^2 t\right]^2 = \left[\frac{1}{2}(1 - \cos 2t)\right]^2 = \frac{1}{4}(1 - 2 \cos 2t + \cos^2 2t).$$

It then follows that

$$\cos^4 t + \sin^4 t = \frac{1}{4}(2 + 2 \cos^2 2t)$$

$$= \frac{1}{2}(1 + \cos^2 2t)$$

$$= \frac{1}{2} \left[ 1 + \frac{1}{2}(1 + \cos 4t) \right]$$

$$= \frac{3}{4} + \frac{1}{4} \cos 4t.$$
It then follows that
\[
\int_C x^3 \, dy - y^3 \, dx = \int_0^{2\pi} \left( \frac{3}{4} + \frac{1}{4} \cos 4t \right) \, dt = \frac{3}{4} (2\pi) = \frac{3\pi}{2}.
\]
\[
\square
\]

2. Let \( F(x, y) = y \hat{i} - x \hat{j} \) and \( R \) be the square in the \( xy \)-plane with vertices \((0, 0), (2, -1), (3, 1) \) and \((1, 2)\). Evaluate \( \int_{\partial R} F \cdot n \, ds \).

**Solution:** Apply the divergence form of Green’s theorem,
\[
\int_{\partial R} F \cdot n \, ds = \iint_R \text{div} F \, dx \, dy,
\]
where
\[
\text{div} F = \frac{\partial}{\partial x} (y) + \frac{\partial}{\partial y} (-x) = 0,
\]
we see that
\[
\int_{\partial R} F \cdot n \, ds = 0.
\]
\[
\square
\]

3. Consider the iterated integral
\[
\int_0^1 \int_y^1 e^{-x^2} \, dx \, dy.
\]

(a) Identify the region of integration, \( R \), for this integral and sketch it.

**Solution:** \( R = \{ (x, y) \in \mathbb{R}^2 \mid y \leq x \leq 1, 1 \leq y \leq 1 \} \) is pictured in Figure 1.

(b) Change the order of integration in the iterated integral and evaluate the double integral
\[
\int_R e^{-x^2} \, dx \, dy.
\]
Figure 1: Region $R$

**Solution:** Compute

$$\int_R e^{-x^2} \, dx \, dy = \int_0^1 \int_0^x e^{-x^2} \, dy \, dx$$

$$= \int_0^1 xe^{-x^2} \, dx$$

$$= \left[ -\frac{1}{2}e^{-x^2} \right]_0^1$$

$$= \frac{1}{2} \left( 1 - \frac{1}{e} \right).$$

4. What is the region $R$ over which you integrate when evaluating the double integral

$$\int_0^1 \int_{x^2}^1 x\sqrt{1-y^2} \, dy \, dx?$$

Rewrite this as an iterated integral first with respect to $x$, then with respect to $y$. Evaluate this integral. Which order of integration is easier?

**Solution:** The region $R = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, x^2 \leq y \leq 1\}$ is sketched on Figure 2m on page 4.
\[ \int_R x \sqrt{1 - y^2} \, dx \, dy = \int_0^1 \int_0^{\sqrt{y}} x \sqrt{1 - y^2} \, dx \, dy \]

\[ = \int_0^1 \left[ \sqrt{1 - y^2} - \frac{1}{2} x^2 \right]_0^{\sqrt{y}} \, dy \]

\[ = \frac{1}{2} \int_0^1 \sqrt{1 - y^2} \, dy \]

\[ = -\frac{1}{4} \int_0^1 \sqrt{1 - y^2} (-2y) \, dy \]

\[ = -\frac{1}{4} \int_1^0 \sqrt{u} \, du \]

\[ = \frac{1}{4} \int_0^1 \sqrt{u} \, du \]

\[ = \frac{1}{6}. \]

\[ \square \]

5. What is the region \( R \) over which you integrate when evaluating the double integral

\[ \int_1^2 \int_1^x \frac{x}{\sqrt{x^2 + y^2}} \, dy \, dx? \]
Rewrite this as an iterated integral first with respect to $x$, then with respect to $y$. Evaluate this integral. Which order of integration is easier?

**Solution:** The region $R = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 2, 1 \leq y \leq x\}$ is sketched on Figure 3 on page 5.

![Figure 3: Region R](image)

\[
\int_{R} \frac{x}{\sqrt{x^2 + y^2}} \, dy \, dx = \int_{1}^{2} \int_{y}^{1} \frac{x}{\sqrt{x^2 + y^2}} \, dx \, dy
\]

\[
= \frac{1}{2} \int_{1}^{2} \int_{y}^{1} \frac{2x}{\sqrt{x^2 + y^2}} \, dx \, dy
\]

\[
= \int_{1}^{2} \int_{2y^2}^{1+y^2} \frac{1}{2\sqrt{u}} \, du \, dy
\]

\[
= \int_{1}^{2} \left( \sqrt{1+y^2} - \sqrt{2} y \right) \, dy
\]

\[
= \int_{1}^{2} \sqrt{1+y^2} \, dy - \frac{3\sqrt{2}}{2},
\]

where

\[
\int_{1}^{2} \sqrt{1+y^2} \, dy = \left[ \frac{1}{2} \left( y\sqrt{1+y^2} + \ln |y + \sqrt{1+y^2}| \right) \right]_{1}^{2}
\]

\[
= \sqrt{5} - \frac{\sqrt{2}}{2} + \frac{1}{2} \ln \left( \frac{2 + \sqrt{5}}{1 + \sqrt{2}} \right).
\]
It then follows that
\[ \int_R \frac{x}{\sqrt{x^2 + y^2}} \, dy \, dx = \sqrt{5} - 2\sqrt{2} + \frac{1}{2} \ln \left( \frac{2 + \sqrt{5}}{1 + \sqrt{2}} \right). \]