Solutions to Assignment #1

1. Let \( \vec{v}_1 = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \) and \( \vec{v}_2 = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} \).

(a) Give the parametric equations of the line through the point \( P: (0, 4, 7) \) in the direction of the vector \( \vec{v}_1 \).

(b) Give the equation of the plane through the point \( P: (0, 4, 7) \) and perpendicular to a direction which is perpendicular to both vectors \( \vec{v}_1 \) and \( \vec{v}_2 \).

**Solution:**

\[
(a) \begin{cases}
  x = -t \\
  y = 4 + 2t \\
  z = 7 - 2t
\end{cases}
\]

(b) The plane is given by

\[
\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 7 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} + s \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}, \; t, s \in \mathbb{R} \right\}.
\]

This leads to the three equations

\[
\begin{cases}
  x = -t + 3s \\
  y = 4 + 2t - 5s \\
  z = 7 - 2t + 4s
\end{cases}
\]

which can be expressed as a system of linear equations in which the unknowns are \( t \) and \( s \):

\[
\begin{aligned}
-t + 3s &= x \\
2t - 5s &= y - 4 \\
-2t + 4s &= z - 7
\end{aligned}
\]

We can use Gaussian elimination to determine conditions on \( x \), \( y \) and \( z \) for which it is possible to solve this system for \( t \) and \( s \):

\[
\begin{pmatrix}
-1 & 3 & | & x \\
2 & -5 & | & y - 4 \\
-2 & 4 & | & z - 7
\end{pmatrix} \to \begin{pmatrix}
1 & -3 & | & -x \\
2 & -5 & | & y - 4 \\
-2 & 4 & | & z - 7
\end{pmatrix} \to
\]

\[
\begin{pmatrix}
1 & -3 & | & -x \\
2 & -5 & | & y - 4 \\
-2 & 4 & | & z - 7
\end{pmatrix}
\]
Thus, for the system to have a solution, we must have that

\[ 2x + 2(y - 4) + (z - 7) = 0. \]

This is the equation of the plane, which may be simplified to

\[ 2x + 2y + z = 15. \]

\[ \square \]

2. The following give parametric equations to two lines in \( \mathbb{R}^3 \):

\[
\begin{align*}
    x &= -1 + 4t \\
    y &= -7t \\
    z &= 2 - t
\end{align*}
\]

\[
\begin{align*}
    x &= -1 + s \\
    y &= 2 - s \\
    z &= 2s
\end{align*}
\]

Determine if the two lines ever meet. Justify your answer. If the lines do meet, give the equation of the plane that contains both lines.

**Solution:** Suppose there are values for \( t \) and \( s \) that yield the same point in space. Then,

\[
\begin{align*}
    -1 + 4t &= -1 + s \\
    -7t &= 2 - s \\
    2 - t &= 2s
\end{align*}
\]

from which we get the system

\[
\begin{align*}
    4t - s &= 0 \\
    -7t + s &= 2 \\
    -t - 2s &= -2
\end{align*}
\]

Solving this system by Gaussian elimination we get

\[
\begin{pmatrix}
    4 & -1 & 0 \\
    -7 & 1 & 2 \\
    -1 & -2 & -2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
    1 & 2 & 2 \\
    -7 & 1 & 2 \\
    4 & -1 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
    1 & 2 & 2 \\
    0 & 5 & 0 \\
    0 & 0 & 0
\end{pmatrix}
\]

\[ \square \]
\[
\begin{pmatrix}
1 & 2 & | & 2 \\
0 & 15 & | & 16 \\
0 & -9 & | & -8
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & | & 2 \\
0 & 1 & | & \frac{16}{15} \\
0 & 0 & | & \frac{8}{5}
\end{pmatrix}
\]

The last row yields a contradiction. Thus, the system has no solutions and therefore the lines do not meet. \(\square\)

3. The following give parametric equations to two lines in \(\mathbb{R}^3\):

\[
\begin{align*}
\begin{cases}
x &= 2 + 4t \\
y &= -1 - 7t \\
z &= 2 - t
\end{cases}
\quad \begin{cases}
x &= s \\
y &= 1 - s \\
z &= -2 + 2s
\end{cases}
\end{align*}
\]

Determine if the two lines ever meet. Justify your answer. If the lines do meet, give the equation of the plane that contains both lines.

**Solution:** Proceeding as in the previous problem (or by inspection), we find that when \(t = 0\) and \(s = 2\) the two lines yield the same point \((2, -1, 2)\). Thus, the two lines do meet at that point.

The two lines are contained in a plane through that point spanned by the direction vectors of the two lines

\[
\begin{pmatrix}
4 \\
-7 \\
-1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 \\
-1 \\
2
\end{pmatrix}.
\]

Thus, the plane is given by

\[
\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 4 \\ -7 \\ -1 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \ t, s \in \mathbb{R} \right\}.
\]

Proceeding as in the solution to problem 1, we are lead to the solving the system that yields the augmented matrix

\[
\begin{pmatrix}
4 & 1 & | & x - 2 \\
-7 & -1 & | & y + 1 \\
-1 & 2 & | & z - 2
\end{pmatrix}
\]

Performing Gaussian elimination yields

\[
\begin{pmatrix}
1 & -2 & | & -(2 - 2) \\
0 & 1 & | & -\frac{1}{15}(y + 1) + \frac{7}{15}(z - 2) \\
0 & 0 & | & (x - 2) + \frac{3}{5}(y + 1) - \frac{1}{5}(z - 2)
\end{pmatrix}
\]
It then follows that the equation of the plane containing the two lines is

\[(x - 2) + \frac{3}{5}(y + 1) - \frac{1}{5}(z - 2) = 0\]

or

\[5(x - 2) + 3(y + 1) - (z - 2) = 0\]

or

\[5x + 3y - z = 5.\]

\[\square\]

4. The vectors \(\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\) and \(\vec{v}_3 = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}\) in \(\mathbb{R}^3\) can span a line, a plane or the entire three dimensional space \(\mathbb{R}^3\). Give the equation of the geometric object which they span.

**Solution:** Place the vectors as rows in the matrix

\[
\begin{pmatrix}
-1 & 1 & 2 \\
1 & 0 & -1 \\
3 & 4 & 1
\end{pmatrix}
\]

and perform elementary row operations, keeping a record of the changes, to obtain that

\[
\begin{pmatrix}
-1 & 1 & 2 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

where the last row was obtained by \(-4\vec{v}_1 - 7\vec{v}_2 + \vec{v}_3\). It then follows that the three vectors are linearly dependent with the third one expressed as

\[v_3 = 4\vec{v}_1 + 7\vec{v}_2,\]

the first two vectors being linearly independent. Consequently, the three vectors generate a plane spanned by the first two vectors. Proceeding as in problems 1 and 3, we find that the equation of the plane is

\[x - y + z = 0.\]

\[\square\]
5. *(Exercises 10 and 11 on page 50 in the text.)* Consider the plane whose equation is
\[ x - 4y + 7z = 3 \]

10. Find a vector perpendicular to this plane.

*Solution:* \( n = \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix} \). \( \square \)

11. Find two vectors that span this plane.

*Solution:* The points \( P_0(3, 0, 0) \), \( P_1(0, -3/4, 0) \) and \( P_2(0, 0, 3/7) \) are three points on the plane. The vectors
\[ \vec{v}_1 = \overrightarrow{P_0P_1} = \begin{pmatrix} -3 \\ -\frac{3}{4} \\ 0 \end{pmatrix} \]
\[ \vec{v}_2 = \overrightarrow{P_0P_2} = \begin{pmatrix} -3 \\ 0 \\ \frac{3}{7} \end{pmatrix} \]
lie on the plane and they also generate it. \( \square \)