Assignment #2

Due on Monday, September 14, 2009

Read Chapter 2 on *Vector Algebra* in Bressoud (pp. 29–49).

Do the following problems

1. Recall that the dot product, or inner product, of two vectors in $\mathbb{R}^n$ is symmetric, bi–linear and positive definite; that is, for vectors v, v_1, v_2 and w in $\mathbb{R}^n$,

   (i) $v \cdot w = w \cdot v$

   (ii) $(c_1 v_1 + c_2 v_2) \cdot w = c_1 v_1 \cdot w + c_2 v_2 \cdot w$, and

   (iii) $v \cdot v \geqslant 0$ for all $v \in \mathbb{R}^n$ and $v \cdot v = 0$ if and only if v is the zero vector.

   Use these properties of the the inner product in $\mathbb{R}^n$ to derive the following properties of the norm $\|\cdot\|$ in $\mathbb{R}^n$, where

   $$\|v\| = \sqrt{v \cdot v} \quad \text{for all vectors} \quad v \in \mathbb{R}^n.$$

   (a) $\|v\| \geqslant 0$ for all $v \in \mathbb{R}^n$ and $\|v\| = 0$ if and only if $v = \vec{0}$.

   (b) For a scalar c, $\|cv\| = |c|\|v\|$.

2. Recall the Cauchy-Schwarz inequality: For any vectors v and w in $\mathbb{R}^n$,

   $$|v \cdot w| \leqslant \|v\|\|w\|.$$

   Use this inequality to derive the triangle inequality: For any vectors v and w in $\mathbb{R}^n$,

   $$\|v + w\| \leqslant \|v\| + \|w\|.$$

   (*Suggestion:* Start with the expression $\|v + w\|^2$ and use the properties of the inner product to simplify it.)

3. Given two non–zero vectors v and w in $\mathbb{R}^n$, the cosine of the angle, θ, between the vectors can be defined by

   $$\cos\theta = \frac{v \cdot w}{\|v\|\|w\|}.$$

   Use the Cauchy-Schwarz inequality to justify why this definition makes sense.

4. Two vectors v and w in $\mathbb{R}^n$ are said to be *orthogonal* or perpendicular, if and only if $v \cdot w = 0$.

   Show that if v and w are orthogonal, then

   $$\|v + w\|^2 = \|v\|^2 + \|w\|^2.$$

   Give a geometric interpretation of this result in two–dimensional Euclidean space.

5. A vector u in $\mathbb{R}^n$ is said to be a unit vector if and only if $\|u\| = 1$. Let u be a unit vector in $\mathbb{R}^n$ and v be any vector in $\mathbb{R}^n$.

   (a) Give the parametric equation of the line through origin in the direction of u.

   (b) Let $f(t) = \|v - tu\|^2$ for all $t \in \mathbb{R}^n$. Explain why this function gives the square of the distance from the point at v to a point on the line through the origin in the direction of u.

   (c) Show that $f(t)$ is minimized when $t = v \cdot u$.