Assignment #2

Due on Monday, September 14, 2009

Read Chapter 2 on Vector Algebra in Bressoud (pp. 29–49).

Do the following problems

1. Recall that the dot product, or inner product, of two vectors in $\mathbb{R}^n$ is symmetric, bi–linear and positive definite; that is, for vectors $v$, $v_1$, $v_2$ and $w$ in $\mathbb{R}^n$,

   (i) $v \cdot w = w \cdot v$
   
   (ii) $(c_1 v_1 + c_2 v_2) \cdot w = c_1 v_1 \cdot w + c_2 v_2 \cdot w$, and
   
   (iii) $v \cdot v \geq 0$ for all $v \in \mathbb{R}^n$ and $v \cdot v = 0$ if and only if $v$ is the zero vector.

   Use these properties of the inner product in $\mathbb{R}^n$ to derive the following properties of the norm $\| \cdot \|$ in $\mathbb{R}^n$, where $\|v\| = \sqrt{v \cdot v}$ for all vectors $v \in \mathbb{R}^n$.

   (a) $\|v\| \geq 0$ for all $v \in \mathbb{R}^n$ and $\|v\| = 0$ if and only if $v = \vec{0}$.
   
   (b) For a scalar $c$, $\|cv\| = |c|\|v\|.$

2. Recall the Cauchy-Schwarz inequality: For any vectors $v$ and $w$ in $\mathbb{R}^n$,

   \[ |v \cdot w| \leq \|v\| \|w\|. \]

   Use this inequality to derive the triangle inequality: For any vectors $v$ and $w$ in $\mathbb{R}^n$,

   \[ \|v + w\| \leq \|v\| + \|w\|. \]

   (Suggestion: Start with the expression $\|v + w\|^2$ and use the properties of the inner product to simplify it.)

3. Given two non–zero vectors $v$ and $w$ in $\mathbb{R}^n$, the cosine of the angle, $\theta$, between the vectors can be defined by

   \[ \cos \theta = \frac{v \cdot w}{\|v\| \|w\|}. \]

   Use the Cauchy-Schwarz inequality to justify why this definition makes sense.
4. Two vectors $v$ and $w$ in $\mathbb{R}^n$ are said to be orthogonal or perpendicular, if and only if $v \cdot w = 0$.

Show that if $v$ and $w$ are orthogonal, then

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2.$$ 

Give a geometric interpretation of this result in two–dimensional Euclidean space.

5. A vector $u$ in $\mathbb{R}^n$ is said to be a unit vector if and only if $\|u\| = 1$. Let $u$ be a unit vector in $\mathbb{R}^n$ and $v$ be any vector in $\mathbb{R}^n$.

(a) Give the parametric equation of the line through origin in the direction of $u$.

(b) Let $f(t) = \|v - tu\|^2$ for all $t \in \mathbb{R}^n$. Explain why this function gives the square of the distance from the point at $v$ to a point on the line through the origin in the direction of $u$.

(c) Show that $f(t)$ is minimized when $t = v \cdot u$. 