

Assignment #6

Due on Monday, September 28, 2009

Read Section 7.1 on *Limits*, pp. 171–178, in Bressoud.

Background and Definitions

- (*Continuous Function*) Let U denote an open subset of \mathbb{R}^n . A function $F: U \rightarrow \mathbb{R}^m$ is said to be continuous at $x \in U$ if and only if $\lim_{\|y-x\| \rightarrow 0} \|F(y) - F(x)\| = 0$.
- (*Image*) If $A \subseteq U$, the *image* of A under the map $F: U \rightarrow \mathbb{R}^m$, denoted by $F(A)$, is defined as the set $F(A) = \{y \in \mathbb{R}^m \mid y = F(x) \text{ for some } x \in A\}$.
- (*Pre-image*) If $B \subseteq \mathbb{R}^m$, the *pre-image* of B under the map $F: U \rightarrow \mathbb{R}^m$, denoted by $F^{-1}(B)$, is defined as the set $F^{-1}(B) = \{x \in U \mid F(x) \in B\}$.

Note that $F^{-1}(B)$ is always defined even if F does not have an inverse map.

Do the following problems

1. Use the triangle inequality to prove that, for any x and y in \mathbb{R}^n ,

$$|\|y\| - \|x\|| \leq \|y - x\|.$$

Use this inequality to deduce that the function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$f(x) = \|x\| \quad \text{for all } x \in \mathbb{R}^n$$

is continuous on \mathbb{R}^n .

2. Let $f(x, y)$ and $g(x, y)$ denote two functions defined on a open region, D , in \mathbb{R}^2 . Prove that the vector field $F: D \rightarrow \mathbb{R}^2$, defined by

$$F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

is continuous on D if and only if f and g are both continuous on D .

3. Let U denote an open subset of \mathbb{R}^n and let $F: U \rightarrow \mathbb{R}^m$ and $G: U \rightarrow \mathbb{R}^m$ be two given functions.

- (a) Explain how the sum $F + G$ is defined.
- (b) Prove that if both F and G are continuous on U , then their sum is also continuous.
(*Suggestion:* The triangle inequality might come in handy.)
4. In each of the following, given the function $F: U \rightarrow \mathbb{R}^m$ and the set B , compute the pre-image $F^{-1}(B)$.

(a) $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y^2 \\ x^2 - y^2 \end{pmatrix}$, and $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

(b) $f: D' \rightarrow \mathbb{R}$,

$$f(x, y) = \frac{1}{\sqrt{1 - x^2 - y^2}}, \quad \text{for } (x, y) \in D'$$

where $D' = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 1\}$ (the punctured unit disc),
 $B = \{1\}$.

- (c) $f: D' \rightarrow \mathbb{R}$ is as in part (b), and $B = \{2\}$.
- (d) $f: D' \rightarrow \mathbb{R}$ is as in part (b), and $B = \{1/2\}$.

5. Compute the image the given sets under the following maps

- (a) $\sigma: \mathbb{R} \rightarrow \mathbb{R}^2$, $\sigma(t) = (\cos t, \sin t)$ for all $t \in \mathbb{R}$. Compute $\sigma(\mathbb{R})$.
- (b) $f: D' \rightarrow \mathbb{R}$ and D' are as given in part (b) of the previous problem.
Compute $f(D')$.