Exam 1

September 30, 2009

Name: ______________________________

This is a closed book exam. Show all significant work and justify all your answers. Use your own paper and/or the paper provided by the instructor. You have 75 minutes to work on the following 7 problems. Relax.

1. The points $P(1,0,0)$, $Q(0,2,0)$ and $R(0,0,3)$ determine a unique plane in three dimensional Euclidean space, $\mathbb{R}^3$.

   (a) Give the equation of the plane determined by P, Q and R.

   (b) Give the parametric equations of the line through the point $(1,1,1)$ which is orthogonal to the plane determined by P, Q and R.

   (c) Find the intersection between the line found in part (b) above and the plane determined by P, Q and R.

2. Let P, Q and R be the points given in Problem 1.

   (a) Give the coordinates of the point in the plane determined by P, Q and R which is the closest to the point $(1,1,1)$.

   (b) Find the (shortest) distance from the point $(1,1,1)$ to the plane determined by P, Q and R.

3. Let P, Q and R be the points given in Problem 1.

   Give the area of the triangle whose vertices are P, Q and R.

4. Let U denote an open subset of $\mathbb{R}^n$, and let $F\colon U \to \mathbb{R}^m$ be a vector valued function defined on U.

   (a) State precisely what it means for F to be continuous at $u \in U$.

   (b) Assume that there is a constant $K \geqslant 0$ such that

   $$\|F(v_1) - F(v_2)\| \leqslant K\|v_1 - v_2\| \quad \text{for all } v_1, v_2 \in U. \qquad (1)$$

   Prove that F is continuous on U.

5. Given $w \in \mathbb{R}^n$, define $f\colon \mathbb{R}^n \to \mathbb{R}$ by
$$f(v) = w \cdot v \quad \text{for all } v \in \mathbb{R}^n;$$
that is, $f(v)$ is the dot product of w with v.

   (a) Use the Cauchy–Schwarz inequality to verify that f satisfies the condition (1) in part (b) of Problem 4; namely,
   $$|f(v_1) - f(v_2)| \leqslant K\|v_1 - v_2\| \quad \text{for all } v_1, v_2 \in \mathbb{R}^n.$$
   What is K in this case?
   Deduce therefore that f is continuous on $\mathbb{R}^n$.

   (b) Deduce also that the function $P_i(x_1, x_2, \ldots, x_n) = x_i$, for all points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$, is continuous on $\mathbb{R}^n$; where x_i denotes the i^{th} coordinate of the point $(x_1, x_2, \ldots, x_n)$ for $i = 1, 2, \ldots, n$. Explain your reasoning.

6. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$f(x,y) = \begin{cases} \dfrac{|x|y}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0); \\ \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$
Prove that f is continuous at $(0,0)$.

7. Is the function $f\colon \mathbb{R}^2 \to \mathbb{R}$, defined by
$$f(x,y) = \begin{cases} \dfrac{|x|}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0); \\ \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$
continuous at $(0,0)$? Justify your answer.