Solutions to Review Problems for Exam 1

1. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the plane given by
$$4x - y - 3z = 12.$$

Solution: The point $P_o(3, 0, 0)$ is in the plane. Let
$$w = \overrightarrow{P_oP} = \begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix}$$

The vector $n = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$ is orthogonal to the plane. To find the shortest distance, d, from P to the plane, we compute the norm of the orthogonal projection of w onto n; that is,
$$d = \|\mathrm{P}_{\widehat{n}}(w)\|,$$
where
$$\widehat{n} = \frac{1}{\sqrt{26}} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix},$$
a unit vector in the direction of n, and
$$\mathrm{P}_{\widehat{n}}(w) = (w \cdot \widehat{n})\widehat{n}.$$

It then follows that
$$d = |w \cdot \widehat{n}|,$$
where $w \cdot \widehat{n} = \dfrac{1}{\sqrt{26}}(4 + 21) = \dfrac{25}{\sqrt{26}}$. Hence, $d = \dfrac{25\sqrt{26}}{26} \approx 4.9.$ $\square$

2. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the line given by the parametric equations
$$\begin{cases} x = -1 + 4t, \\ y = -7t, \\ z = 2 - t. \end{cases}$$

***Solution**:* The point $P_o(-1,0,2)$ is on the line. The vector

$$v = \begin{pmatrix} 4 \\ -7 \\ -1 \end{pmatrix}$$

gives the direction of the line. Put

$$w = \overrightarrow{P_oP} = \begin{pmatrix} 5 \\ 0 \\ -9 \end{pmatrix}.$$

The vectors v and w determine a parallelogram whose area is the norm of v times the shortest distance, d, from P to the line determined by v at P_o. We then have that

$$\text{area}(P(v,w)) = \|v\|d,$$

from which we get that

$$d = \frac{\text{area}(P(v,w))}{\|v\|}.$$

On the other hand,

$$\text{area}(P(v,w)) = \|v \times w\|,$$

where

$$v \times w = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 4 & -7 & -1 \\ 5 & 0 & -9 \end{vmatrix} = 63\widehat{i} + 31\widehat{j} + 35\widehat{k}.$$

Thus, $\|v \times w\| = \sqrt{(63)^2 + (31)^2 + (35)^2} = \sqrt{6155}$ and therefore

$$d = \frac{\sqrt{6155}}{\sqrt{66}} \approx 9.7.$$

□

3. Compute the area of the triangle whose vertices in $\mathbb{R}^3$ are the points $(1,1,0)$, $(2,0,1)$ and $(0,3,1)$

***Solution**:* Label the points $P_o(1,1,0)$, $P_1(2,0,1)$ and $P_2(0,3,1)$ and define the vectors

$$v = \overrightarrow{P_oP_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad w = \overrightarrow{P_oP_2} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

The area of the triangle determined by the points P_o, P_1 and P_2 is then half of the area of the parallelogram determined by the vectors v and w. Thus,

$$\text{area}(\triangle P_oP_1P_2) = \frac{1}{2}\|v \times w\|,$$

where

$$v \times w = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 1 & -1 & 1 \\ -1 & 2 & 1 \end{vmatrix} = -3\widehat{i} - 2\widehat{j} + \widehat{k}.$$

Consequently, $\text{area}(\triangle P_oP_1P_2) = \dfrac{1}{2}\sqrt{9+4+1} = \dfrac{\sqrt{14}}{2} \approx 1.87.$ □

4. Let v and w be two vectors in $\mathbb{R}^3$, and let λ be a scalar. Show that the area of the parallelogram determined by the vectors v and $w + \lambda v$ is the same as that determined by v and w.

***Solution**:* The area of the parallelogram determined by v and $w+\lambda v$ is

$$\text{area}(P(v, w+\lambda v)) = \|v \times (w + \lambda v)\|,$$

where

$$v \times (w + \lambda v) = v \times w + \lambda v \times v = v \times w.$$

Consequently, $\text{area}(P(v, w+\lambda v)) = \|v \times w\| = \text{area}(P(v,w)).$ □

5. Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$ and $P_{\widehat{u}}(v)$ denote the orthogonal projection of v along the direction of $\widehat{u}$ for any vector $v \in \mathbb{R}^n$. Use the Cauchy–Schwarz inequality to prove that the map

$$v \mapsto P_{\widehat{u}}(v) \quad \text{for all } \ v \in \mathbb{R}^n$$

is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

***Solution**:* $P_{\widehat{u}}(v) = (v \cdot \widehat{u})\widehat{u}$ for all $v \in \mathbb{R}^n$. Consequently, for any $w, v \in \mathbb{R}^n$,

$$\begin{array}{rcl} P_{\widehat{u}}(w) - P_{\widehat{u}}(v) & = & (w \cdot \widehat{u})\widehat{u} - (v \cdot \widehat{u})\widehat{u} \\ & = & (w \cdot \widehat{u} - v \cdot \widehat{u})\widehat{u} \\ & = & [(w - v) \cdot \widehat{u}]\widehat{u}. \end{array}$$

It then follows that

$$\|P_{\widehat{u}}(w) - P_{\widehat{u}}(v)\| = |(w - v) \cdot \widehat{u}|,$$

since $\|\widehat{u}\| = 1$. Hence, by the Cauchy–Schwarz inequality,

$$\|P_{\widehat{u}}(w) - P_{\widehat{u}}(v)\| \leqslant \|w - v\|.$$

Applying the Squeeze Theorem we then get that

$$\lim_{\|w-v\|\to 0} \|P_{\widehat{u}}(w) - P_{\widehat{u}}(v)\| = 0,$$

which shows that $P_{\widehat{u}}$ is continuous at every $v \in V$. □

6. Let $U \subseteq \mathbb{R}^n$ be open and $F\colon U \to \mathbb{R}^m$ be function satisfying

$$\|F(v) - F(w)\| \leqslant K\|v - w\|^\alpha \quad \text{for all } v, w \in U, \tag{1}$$

and some positive constants K and α.

Prove that F is continuous on U.

***Solution**:* Let u be any vector in u. Then, since U is open, there exists $r > 0$ such that $B_r(u) \subseteq U$. By the condition in (1), for any $v \in B_r(u)$,

$$0 \leqslant \|F(v) - F(u)\| \leqslant K\|v - u\|^\alpha.$$

Now, since $\alpha > 0$,

$$\lim_{\|v-u\|\to 0} \|v - u\|^\alpha = 0.$$

Consequently, by the Squeeze Theorem,

$$\lim_{\|v-u\|\to 0} \|F(v) - F(u)\| = 0,$$

which shows that F is continuous at u. Since u was an arbitrary element of U, we have shown that F is continuous on U. □

7. Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$$

Prove that f is continuous at $(0,0)$.

***Solution**:* For $(x,y)\neq(0,0)$

$$\begin{aligned} |f(x,y)| &= \frac{x^2|y|}{x^2+y^2} \\ &\leqslant |y| \\ &\leqslant \sqrt{x^2+y^2}. \end{aligned}$$

We then have that, for $(x,y)\neq(0,0)$,

$$0 \leqslant |f(x,y)-f(0,0)| \leqslant \|(x,y)-(0,0)\|.$$

Thus, by the Squeeze Theorem,

$$\lim_{\|(x,y)-(0,0)\|\to 0} |f(x,y)-f(0,0)| = 0,$$

which shows that f is continuous at $(0,0)$. □

8. Show that

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

is not continuous at $(0,0)$.

***Solution**:* Let $\varepsilon = \dfrac{1}{2}$ and observe that for any $\delta > 0$

$$f\left(\frac{\delta}{2}, 0\right) = 1.$$

Thus,

$$\left\|\left(\frac{\delta}{2}, 0\right)\right\| = \frac{\delta}{2} < \delta,$$

but
$$\left| f\left(\frac{\delta}{2}, 0\right) - f(0,0)\right| = 1 > \frac{1}{2} = \varepsilon.$$
Hence, f is not continuous at $(0,0)$. □

9. Determine the value of L that would make the function
$$f(x,y) = \begin{cases} x\ \sin\left(\dfrac{1}{y}\right) & \text{if } \ y \neq 0; \\ \\ L & \text{otherwise }, \end{cases}$$
continuous at $(0,0)$. Is $f\colon \mathbb{R}^2 \to \mathbb{R}$ continuous on $\mathbb{R}^2$? Justify your answer.

***Solution**:* Observe that, for $y \neq 0$,
$$\begin{aligned} |f(x,y)| &= \left| x\ \sin\left(\frac{1}{y}\right)\right| \\ &= |x| \left| \sin\left(\frac{1}{y}\right)\right| \\ &\leqslant |x| \\ &\leqslant \sqrt{x^2+y^2}. \end{aligned}$$
It then follows that, for $y \neq 0$,
$$0 \leqslant |f(x,y)| \leqslant \|(x,y)\|.$$
Consequently, by the Squeeze Theorem,
$$\lim_{\|(x,y)\|\to 0} |f(x,y)| = 0.$$
This suggests that we define $L = 0$. If this is the case,
$$\lim_{\|(x,y)\|\to 0} |f(x,y) - f(0,0)| = 0,$$
which shows that f is continuous at $(0,0)$ if $L = 0$.

Assume now that $L = 0$ in the definition of f. Then, for any $a \neq 0$, f fails for be continuous at $(a, 0)$. To see why this is case, note that for any $y \neq 0$

$$|f(a, y)| = |a| \left|\sin\left(\frac{1}{y}\right)\right|$$

and the limit of $\left|\sin\left(\frac{1}{y}\right)\right|$ as $y \to 0$ does not exist. □

10. Define $G \colon \mathbb{R}^2 \to \mathbb{R}$ by $G(x, y) = xy$ for all $(x, y) \in \mathbb{R}^2$. Prove that G is continuous on $\mathbb{R}^2$; that is, prove that

$$\lim_{(x,y)\to(x_o,y_o)} G(x, y) = G(x_o, y_o) \quad \text{for all} \quad (x_o, y_o) \in \mathbb{R}^2$$

or

$$\lim_{(x,y)\to(x_o,y_o)} |G(x, y) - G(x_o, y_o)| = 0 \quad \text{for all} \quad (x_o, y_o) \in \mathbb{R}^2.$$

Proof: Using the triangle inequality we obtain

$$\begin{aligned} |G(x, y) - G(x_o, y_o)| &= |xy - x_o y_o| \\ &= |xy - x_o y + x_o y - x_o y_o| \\ &= |(x - x_o)y + x_o(y - y_o)| \\ &\leqslant |x - x_o|\ |y| + |x_o|\ |y - y_o|. \end{aligned}$$

Next, use the estimates

$$|x - x_o| \leqslant \|(x, y) - (x_o, y_o)\|$$

and

$$|y - y_o| \leqslant \|(x, y) - (x_o, y_o)\|$$

to obtain

$$|G(x, y) - G(x_o, y_o)| \leqslant \|(x, y) - (x_o, y_o)\|\ |y| + |x_o|\ \|(x, y) - (x_o, y_o)\|,$$

or

$$|G(x, y) - G(x_o, y_o)| \leqslant (|y| + |x_o|)\ \|(x, y) - (x_o, y_o)\|.$$

Observe that

$$\lim_{(x,y)\to(x_o,y_o)} |y| = |y_o|,$$

which follows from the fact that the map $(x, y) \to y$ is continuous since it is a projection. Thus,

$$\lim_{(x,y)\to(x_o,y_o)} (|y| + |x_o|)\ \|(x, y) - (x_o, y_o)\| = (|y_o| + |x_o|) \cdot 0 = 0,$$

Hence, from

$$0 \leqslant |G(x, y) - G(x_o, y_o)| \leqslant (|y| + |x_o|)\ \|(x, y) - (x_o, y_o)\|,$$

and the Sandwich theorem, it follows that

$$\lim_{(x,y)\to(x_o,y_o)} |G(x, y) - G(x_o, y_o)| = 0,$$

which was to be shown. □

11. Let U denote an open subset of $\mathbb{R}^2$ and let $g\colon U \to \mathbb{R}$ be two scalar fields on U. Assume that $g(x_o, y_o) \neq 0$ for some $(x_o, y_o) \in U$. Prove that if g is continuous at (x_o, y_o), then there exists $\delta > 0$ such that $B_\delta(x_o, y_o) \subseteq U$ and

$$(x, y) \in B_\delta(x_o, y_o) \Rightarrow |g(x, y)| > \frac{|g(x_o, y_o)|}{2}.$$

Suggestion: Consider $\varepsilon = \dfrac{|g(x_o, y_o)|}{2} > 0$.

Solution: Since g is continuous at (x_o, y_o), given $\varepsilon > 0$, there exists $\delta > 0$ such that $B_\delta(x_o, y_o) \subseteq U$ and

$$(x, y) \in B_\delta(x_o, y_o) \Rightarrow |g(x, y) - g(x_o, y_o)| < \varepsilon.$$

Taking $\varepsilon = \dfrac{|g(x_o, y_o)|}{2} > 0$, we get a δ such that $B_\delta(x_o, y_o) \subseteq U$ and

$$(x, y) \in B_\delta(x_o, y_o) \Rightarrow |g(x, y) - g(x_o, y_o)| < \frac{|g(x_o, y_o)|}{2}.$$

Thus, by the triangle inequality,

$$|g(x_o, y_o)| = |g(x_o, y_o) - g(x, y) + g(x, y)| \leqslant |g(x, y) - g(x_o, y_o)| + |g(x, y)|.$$

It then follows that, if $(x, y) \in B_\delta(x_o, y_o)$, then

$$|g(x_o, y_o)| < \frac{|g(x_o, y_o)|}{2} + |g(x, y)|,$$

from which we get that

$$(x, y) \in B_\delta(x_o, y_o) \Rightarrow |g(x, y)| > \frac{|g(x_o, y_o)|}{2}.$$

□

12. Let U, g and (x_o, y_o) be as in the previous problem. Assume that $g(x_o, y_o) \neq 0$ and that g is continuous at (x_o, y_o). Put

$$h(x, y) = \frac{1}{g(x, y)}.$$

Prove that h is continuous at (x_o, y_o).

Suggestion: Use the result of the previous problem and the Squeeze Theorem.

Solution: First observe that, since $g(x_o, y_o) \neq 0$, $h(x_o, y_o)$ is defined. We want to show that

$$\lim_{(x,y)\to(x_o,y_o)} |h(x, y) - h(x_o, y_o)| = \lim_{(x,y)\to(x_o,y_o)} \left| \frac{1}{g(x, y)} - \frac{1}{g(x_o, y_o)} \right| = 0.$$

To show this, compute

$$\left| \frac{1}{g(x, y)} - \frac{1}{g(x_o, y_o)} \right|.$$

Note that if we restrict (x, y) to lie in $B_\delta(x_o, y_o)$, where $\delta > 0$ is as in the previous problem, then

$$|g(x, y)| \geqslant \frac{|g(x_o, y_o)|}{2},$$

by the result of the previous problem. We therefore get that, for $(x, y) \in B_\delta(x_o, y_o)$, $g(x, y) \neq 0$ and

$$\begin{aligned} \left| \frac{1}{g(x, y)} - \frac{1}{g(x_o, y_o)} \right| &= \left| \frac{g(x_o, y_o) - g(x, y)}{g(x, y)g(x_o, y_o)} \right| \\ &= \frac{|g(x, y) - g(x_o, y_o)|}{|g(x, y)|\ |g(x_o, y_o)|} \\ &\leqslant \frac{2}{|g(x_o, y_o)|^2} \cdot |g(x, y) - g(x_o, y_o)|. \end{aligned}$$

Thus, if $(x,y)\in B_\delta(x_o,y_o)$,

$$0\leqslant |h(x,y)-h(x_o,y_o)|\leqslant \frac{2}{|g(x_o,y_o)|^2}\cdot|g(x,y)-g(x_o,y_o)|,$$

where

$$\lim_{(x,y)\to(x_o,y_o)}|g(x,y)-g(x_o,y_o)|=0,$$

since g is continuous at (x_o,y_o). It then follows, by the Sandwich Theorem that

$$\lim_{(x,y)\to(x_o,y_o)}|h(x,y)-h(x_o,y_o)|=0,$$

which was to be shown. □