Review Problems for Exam 2

1. Define the scalar field $f : \mathbb{R}^n \to \mathbb{R}$ by $f(v) = \frac{1}{2} \|v\|^2$ for all $v \in \mathbb{R}^n$. Show that $f$ is differentiable on $\mathbb{R}^n$ and compute the linear map $Df(u) : \mathbb{R}^n \to \mathbb{R}$ for all $u \in \mathbb{R}^n$. What is the gradient of $f$ at $u$ for all $x \in \mathbb{R}^n$?

2. Let $g : [0, \infty) \to \mathbb{R}$ be a differentiable, real–valued function of a single variable, and let $f(x, y) = g(r)$ where $r = \sqrt{x^2 + y^2}$.

   (a) Compute $\frac{\partial r}{\partial x}$ in terms of $x$ and $r$, and $\frac{\partial r}{\partial y}$ in terms of $y$ and $r$.

   (b) Compute $\nabla f$ in terms of $g'(r)$, $r$ and the vector $\mathbf{r} = \hat{x}i + \hat{y}j$.

3. Let $f : U \to \mathbb{R}$ denote a scalar field defined on an open subset $U$ of $\mathbb{R}^n$, and let $\hat{u}$ be a unit vector in $\mathbb{R}^n$. If the limit

$$\lim_{t \to 0} \frac{f(v + t\hat{u}) - f(v)}{t}$$

exists, we call it the directional derivative of $f$ at $v$ in the direction of the unit vector $\hat{u}$. We denote it by $D_{\hat{u}}f(v)$.

   (a) Show that if $f$ is differentiable at $v \in U$, then, for any unit vector $\hat{u}$ in $\mathbb{R}^n$, the directional derivative of $f$ in the direction of $\hat{u}$ at $v$ exists, and

$$D_{\hat{u}}f(v) = \nabla f(v) \cdot \hat{u},$$

where $\nabla f(v)$ is the gradient of $f$ at $v$.

   (b) Suppose that $f : U \to \mathbb{R}$ is differentiable at $v \in U$. Prove that if $D_{\hat{u}}f(v) = 0$ for every unit vector $\hat{u}$ in $\mathbb{R}^n$, then $\nabla f(v)$ must be the zero vector.

   (c) Suppose that $f : U \to \mathbb{R}$ is differentiable at $v \in U$. Use the Cauchy–Schwarz inequality to show that the largest value of $D_{\hat{u}}f(v)$ is $\|\nabla f(v)\|$ and it occurs when $\hat{u}$ is in the direction of $\nabla f(v)$.

4. The scalar field $f : U \to \mathbb{R}$ is said to have a local minimum at $x \in U$ if there exists $r > 0$ such that $B_r(x) \subseteq U$ and

$$f(x) \leq f(y) \text{ for every } y \in B_r(x).$$

Prove that if $f$ is differentiable at $x \in U$ and $f$ has a local minimum at $x$, then $\nabla f(x) = 0$, the zero vector in $\mathbb{R}^n$. 
5. Let $I$ denote an open interval in $\mathbb{R}$. Suppose that $\sigma: I \to \mathbb{R}^n$ and $\gamma: I \to \mathbb{R}^n$ are paths in $\mathbb{R}^n$. Define a real valued function $f: I \to \mathbb{R}$ of a single variable by

$$f(t) = \sigma(t) \cdot \gamma(t) \quad \text{for all} \quad t \in I;$$

that is, $f(t)$ is the dot product of the two paths at $t$.

Show that if $\sigma$ and $\gamma$ are both differentiable on $I$, then so is $f$, and

$$f'(t) = \sigma'(t) \cdot \gamma(t) + \sigma(t) \cdot \gamma'(t) \quad \text{for all} \quad t \in I.$$

6. Let $\sigma: I \to \mathbb{R}^n$ denote a differentiable path in $\mathbb{R}^n$. Show that if $\|\sigma(t)\|$ is constant for all $t \in I$, then $\sigma'(t)$ is orthogonal to $\sigma(t)$ for all $t \in I$.

7. A particle is following a path in three–dimensional space given by

$$\sigma(t) = (e^t, e^{-t}, 1 - t) \quad \text{for} \quad t \in \mathbb{R}.$$

At time $t_o = 1$, the particle flies off on a tangent.

(a) Where will the particle be at time $t_1 = 2$?

(b) Will the particle ever hit the $xy$–plane? Is so, find the location on the $xy$ plane where the particle hits.

8. Let $U$ denote an open and convex subset of $\mathbb{R}^n$. Suppose that $f: U \to \mathbb{R}$ is differentiable at every $x \in U$. Fix $x$ and $y$ in $U$, and define $g: [0, 1] \to \mathbb{R}$ by

$$g(t) = f(x + t(y - x)) \quad \text{for} \quad 0 \leq t \leq 1.$$

(a) Explain why the function $g$ is well defined.

(b) Show that $g$ is differentiable on $(0, 1)$ and that

$$g'(t) = \nabla f(x + t(y - x)) \cdot (y - x) \quad \text{for} \quad 0 < t < 1.$$

(Suggestion: Consider

$$\frac{g(t + h) - g(t)}{h} = \frac{f(x + t(y - x) + h(y - x)) - f(x + t(y - x))}{h}$$

and apply the definition of differentiability of $f$ at the point $x + t(y - x)$.)
(c) Use the Mean Value Theorem for derivatives to show that there exists a point \( z \) is the line segment connecting \( x \) to \( y \) such that

\[
f(y) - f(x) = D_{\hat{u}}f(z)\|y - x\|,
\]

where \( \hat{u} \) is the unit vector in the direction of the vector \( y - x \); that is,

\[
\hat{u} = \frac{1}{\|y - x\|}(y - x).
\]

(Hint: Observe that \( g(1) - g(0) = f(y) - f(x) \).)

9. Prove that if \( U \) is an open and convex subset of \( \mathbb{R}^n \), and \( f: U \to \mathbb{R} \) is differentiable on \( U \) with \( \nabla f(v) = 0 \) for all \( v \in U \), then \( f \) must be a constant function.

10. Let \( f \) be a scalar field defined on \((x,y)\) where \( x = r \cos \theta, y = r \sin \theta \). Show that

\[
\nabla f = \frac{\partial f}{\partial r} \vec{u}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{u}_\theta,
\]

where \( \vec{u}_r = (\cos \theta, \sin \theta) \) and \( \vec{u}_\theta = (-\sin \theta, \cos \theta) \).

(Hint: First find \( \frac{\partial f}{\partial r} \) and \( \frac{\partial f}{\partial \theta} \) in terms of \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) and then solve for \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) int terms of \( \frac{\partial f}{\partial r} \) and \( \frac{\partial f}{\partial \theta} \).

11. Let \( U \) be an open subset of \( \mathbb{R}^n \) and \( I \) be an open interval. Suppose that \( f: U \to \mathbb{R} \) is a differentiable scalar field and \( \sigma: I \to \mathbb{R}^n \) be a differentiable path whose image lies in \( U \). Suppose also that \( \sigma'(t) \) is never the zero vector. Show that if \( f \) has a local maximum or a local minimum at some point on the path, then \( \nabla f \) is perpendicular to the path at that point.

Suggestion: Consider the real valued function of a single variable \( g(t) = f(\sigma(t)) \) for all \( t \in I \).

12. Let \( \sigma: [a, b] \to \mathbb{R}^n \) be a differentiable, one–to–one path. Suppose also that \( \sigma'(t) \), is never the zero vector. Let \( h: [c, d] \to [a, b] \) be a one–to–one and onto map such that \( h'(t) \neq 0 \) for all \( t \in [c, d] \). Define

\[
\gamma(t) = \sigma(h(t)) \quad \text{for all} \quad t \in [c, d].
\]

\( \gamma: [c, d] \to \mathbb{R}^n \) is a called a reparametrization of \( \sigma \)

(a) Show that \( \gamma \) is a differentiable, one–to–one path.
(b) Compute \( \gamma'(t) \) and show that it is never the zero vector.
(c) Show that \( \sigma \) and \( \gamma \) have the same image in \( \mathbb{R}^n \).