Review Problems for Exam 3

1. Consider a wheel of radius $a$ which is rolling on the $x$–axis in the $xy$–plane. Suppose that the center of the wheel moves in the positive $x$–direction and a constant speed $v_0$. Let $P$ denote a fixed point on the rim of the wheel.

(a) Give a path $\sigma(t) = (x(t), y(t))$ giving the position of the $P$ at any time $t$, if $P$ is initially at the point $(0, 2a)$.

(b) Compute the velocity of $P$ at any time $t$. When is the velocity of $P$ horizontal? What is the speed of $P$ at those times?

2. Let $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \geq 0\}$; i.e., $C$ is the upper unit semi–circle. $C$ can be parametrized by

$$\sigma(\tau) = (\tau, \sqrt{1 - \tau^2}) \quad \text{for} \quad -1 \leq \tau \leq 1.$$ 

(a) Compute $s(t)$, the arclength along $C$ from $(-1, 0)$ to the point $\sigma(t)$, for $0 \leq t \leq 1$.

(b) Compute $s'(t)$ for $-1 < t < t$ and sketch the graph of $s$ as function of $t$.

(c) Show that $\cos(\pi - s(t)) = t$ for all $-1 \leq t \leq 1$, and deduce that $\sin(s(t)) = \sqrt{1 - t^2}$ for all $-1 \leq t \leq 1$.

3. Let $C$ denote the unit circle traversed in the counterclockwise direction. Evaluate the line integral $\int_C \frac{x}{2} \ dy - \frac{y}{2} \ dx$.

4. Let $F(x, y) = 2x \hat{i} - y \hat{j}$ and $R$ be the square in the $xy$–plane with vertices $(0, 0)$, $(2, -1)$, $(3, 1)$ and $(1, 2)$. Evaluate $\int_{\partial R} F \cdot n \ ds$.

5. Evaluate the line integral $\int_{\partial R} (x^4 + y) \ dx + (2x - y^4) \ dy$, where $R$ is the rectangular region

$$R = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 3, \ -2 \leq y \leq 1\},$$

and $\partial R$ is traversed in the counterclockwise sense.

6. Integrate the function given by $f(x, y) = xy^2$ over the region, $R$, defined by:

$$R = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, 0 \leq y \leq 4 - x^2\}.$$
7. Let $R$ denote the region in the plane defined by inside of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$  \hspace{1cm} (1)

for $a > 0$ and $b > 0$.

(a) Evaluate the line integral $\int_{\partial R} x \ dy - y \ dx$, where $\partial R$ is the ellipse in (1) traversed in the positive sense.

(b) Use your result from part (a) and the divergence form of Green’s theorem to come up with a formula for computing the area of the region enclosed by the ellipse in (1).

8. Evaluate the double integral $\int_R e^{-x^2} \ dx \ dy$, where $R$ is the region in the $xy$-plane sketched in Figure 1.

![Figure 1: Sketch of Region $R$ in Problem 8](image-url)