Solutions to Review Problems for Exam 3

1. Consider a wheel of radius a which is rolling on the x–axis in the xy–plane. Suppose that the center of the wheel moves in the positive x–direction and a constant speed v_o. Let P denote a fixed point on the rim of the wheel.

   (a) Give a path $\sigma(t) = (x(t), y(t))$ giving the position of the P at any time t, if P is initially at the point $(0, 2a)$.

   Solution: Let $\theta(t)$ denote the angle that the ray from the center

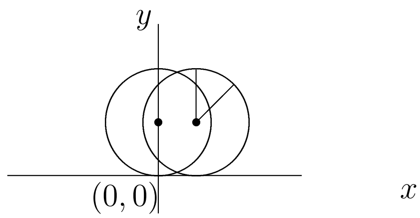

Figure 1: Circle

   of the circle to the point $(x(t), y(t))$ makes with a vertical line through the center. Then, $v_o t = a\theta(t)$; so that $\theta(t) = \dfrac{v_o}{a} t$ and

$$x(t) = v_o t + a\sin(\theta(t))$$

   and

$$y(t) = a + a\cos(\theta(t))$$

□

   (b) Compute the velocity of P at any time t. When is the velocity of P horizontal? What is the speed of P at those times?

   Solution: The velocity vector is

$$\sigma'(t) = (x'(t), y'(t)) = (v_o + a\theta'(t)\cos(\theta(t)), -a\theta'(t)\sin(\theta(t)))$$

   where

$$\theta'(t) = \frac{v_o}{a}.$$

   We then have that

$$\sigma'(t) = (v_o + v_o\cos(\theta(t)), -v_o\sin(\theta(t))).$$

The velocity of P is horizontal when

$$\sin(\theta(t)) = 0,$$

or

$$\theta(t) = n\pi,$$

where n is an integer; and when

$$\cos(\theta(t)) \neq -1.$$

We then get that the velocity of P is horizontal when

$$\theta(t) = 2k\pi$$

where k is an integer.
The speed at the points where the velocity if horizontal is then equal to $2v_o$. □

2. Let $C = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \geqslant 0\}$; i.e., C is the upper unit semi–circle. C can be parametrized by

$$\sigma(\tau) = (\tau, \sqrt{1-\tau^2}) \quad \text{for} \quad -1 \leqslant \tau \leqslant 1.$$

(a) Compute $s(t)$, the arclength along C from $(-1,0)$ to the point $\sigma(t)$, for $0 \leqslant t \leqslant 1$.

Solution: Compute $\sigma'(\tau) = \left(1, -\dfrac{\tau}{\sqrt{1-\tau^2}}\right)$. for all $\tau \in (-1,1)$.
Then,

$$\|\sigma'(\tau)\| = \sqrt{1 + \frac{\tau^2}{1-\tau^2}} = \frac{1}{\sqrt{1-\tau^2}}.$$

It then follows that

$$s(t) = \int_{-1}^{t} \frac{1}{\sqrt{1-\tau^2}}\ \mathrm{d}\tau \quad \text{for} \ -1 \leqslant t \leqslant 1.$$

□

(b) Compute $s'(t)$ for $-1 < t < t$ and sketch the graph of s as function of t.

Solution: By the Fundamental Theorem of Calculus,

$$s'(t) = \frac{1}{\sqrt{1-t^2}} \quad \text{for} \ -1 < t < 1.$$

Note then that $s'(t) > 0$ for all $t \in (-1, 1)$ and therefore s is strictly increasing on $(-1, 1)$.
Next, compute the derivative of $s'(t)$ to get the second derivative of $s(t)$:

$$s''(t) = \frac{t}{(1-t^2)^{3/2}} \quad \text{for } -1 < t < 1.$$

It then follows that $s''(t) < 0$ for $-1 < t < 0$ and $s''(t) > 0$ for $0 < t < 1$. Thus, the graph of $s = s(t)$ is concave down on $(-1, 0)$ and concave up on $(0, 1)$.
Finally, observe that $s(-1) = 0$, $s(0) = \pi/2$ (the arc–length along a quarter of the unit circle), and $s(1) = \pi$ (the arc–length along a semi–circle of unit radius). We can then sketch the graph of $s = s(t)$ as shown in Figure 2. □

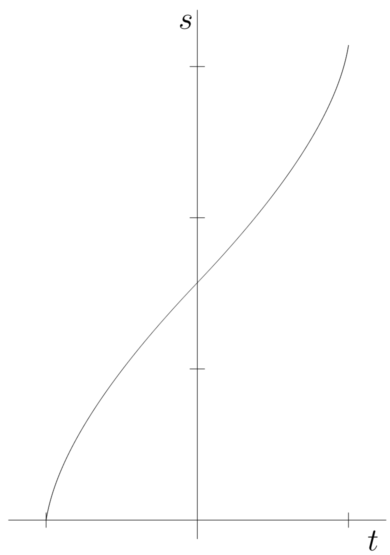

Figure 2: Sketch of $s = s(t)$

(c) Show that $\cos(\pi - s(t)) = t$ for all $-1 \leqslant t \leqslant 1$, and deduce that

$$\sin(s(t)) = \sqrt{1-t^2} \quad \text{for all} \quad -1 \leqslant t \leqslant 1.$$

Solution: Figure 3 shows the upper unit semicircle and a point $\sigma(t)$ on it. Putting $\theta(t) = \pi - s(t)$, then $\theta(t)$ is the angle, in radians, that the ray from the origin to $\sigma(t)$ makes with the positive x–axis. It then follows that

$$\cos(\theta(t)) = t$$

and

$$\sin(\theta(t)) = \sqrt{1-t^2}.$$

Since

$$\sin(\theta(t)) = \sin(\pi - s(t)) = \sin(s(t),$$

the result follows. □

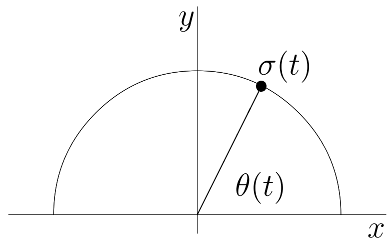

Figure 3: Sketch of Semi–circle

3. Let C denote the unit circle traversed in the counterclockwise direction. Evaluate the line integral $\int_C \frac{x}{2}\ \mathrm{d}y - \frac{y}{2}\ \mathrm{d}x$.

***Solution**:* Let $F(x,y) = \frac{x}{2}\ \widehat{\imath} + \frac{y}{2}\ \widehat{\jmath}$. Then,

$$\int_C \frac{x}{2}\ \mathrm{d}y - \frac{y}{2}\ \mathrm{d}x = \int_C F \cdot \widehat{n}\ \ \mathrm{d}s.$$

Thus, by Green's Theorem in divergence form,

$$\int_C \frac{x}{2}\ \mathrm{d}y - \frac{y}{2}\ \mathrm{d}x = \iint_R \mathrm{div}F\ \ \mathrm{d}x\ \mathrm{d}y,$$

where R is the unit disc bounded by C, and

$$\mathrm{div}F(x,y) = \frac{\partial}{\partial x}\left(\frac{x}{2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{2}\right) = \frac{1}{2} + \frac{1}{2} = 1.$$

Consequently,

$$\int_C \frac{x}{2}\ \mathrm{d}y - \frac{y}{2}\ \mathrm{d}x = \iint_R\ \mathrm{d}x\ \mathrm{d}y = \mathrm{area}(R) = \pi.$$

□

4. Let $F(x,y) = 2x\,\widehat{i} - y\,\widehat{j}$ and R be the square in the xy–plane with vertices $(0,0)$, $(2,-1)$, $(3,1)$ and $(1,2)$. Evaluate $\displaystyle\oint_{\partial R} F\cdot n\ \mathrm{d}s$.

***Solution**:* Apply Green's Theorem in divergence form,

$$\oint_{\partial R} F\cdot n\ \mathrm{d}s = \iint_R \mathrm{div}F\ \ \mathrm{d}x\ \mathrm{d}y,$$

where

$$\mathrm{div}F(x,y) = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-y) = 2 - 1 = 1.$$

Thus,

$$\oint_{\partial R} F\cdot n\ \mathrm{d}s = \iint_R\ \mathrm{d}x\ \mathrm{d}y = \mathrm{area}(R).$$

To find the area of the region R, shown in Figure 4, observe that R is

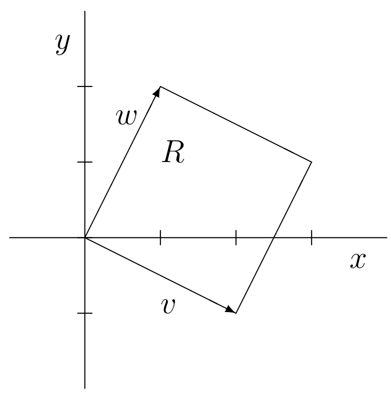

Figure 4: Sketch of Region R in Problem 4

a parallelogram determined by the vectors $v = 2\,\widehat{i} - \widehat{j}$ and $w = \widehat{i} + 2\,\widehat{j}$. Thus,

$$\mathrm{area}(R) = \|v \times w\| = 5.$$

It the follows that

$$\oint_{\partial R} F\cdot n\ \mathrm{d}s = \iint_R\ \mathrm{d}x\ \mathrm{d}y = 5.$$

□

5. Evaluate the line integral $\int_{\partial R} (x^4 + y) \ \ \mathrm{d}x + (2x - y^4) \ \ \mathrm{d}y$, where R is the rectangular region

$$R = \{(x, y) \in \mathbb{R}^2 \mid -1 \leqslant x \leqslant 3, \ -2 \leqslant y \leqslant 1\},$$

and ∂R is traversed in the counterclockwise sense.

Solution: Write

$$\int_{\partial R} (x^4 + y) \ \ \mathrm{d}x + (2x - y^4) \ \ \mathrm{d}y = \int_{\partial R} (2x - y^4) \ \ \mathrm{d}y - [-(x^4 + y)] \ \mathrm{d}x,$$

so that

$$\int_{\partial R} (x^4 + y) \ \ \mathrm{d}x + (2x - y^4) \ \ \mathrm{d}y = \int_{\partial R} F \cdot n \ \ \mathrm{d}s,$$

where F is the vector field

$$F(x, y) = (2x - y^4) \, \widehat{\imath} - (x^4 + y) \, \widehat{\jmath}.$$

Then, by Green's Theorem in divergence form,

$$\int_{\partial R} (x^4 + y) \ \ \mathrm{d}x + (2x - y^4) \ \ \mathrm{d}y = \iint_R \mathrm{div} F \ \ \mathrm{d}x \ \mathrm{d}y,$$

where

$$\mathrm{div} F(x, y) = \frac{\partial}{\partial x}(2x - y^4) - \frac{\partial}{\partial x}(x^4 + y) = 2 - 1 = 1.$$

It then follows that

$$\int_{\partial R} (x^4 + y) \ \ \mathrm{d}x + (2x - y^4) \ \ \mathrm{d}y = \iint_R \ \ \mathrm{d}x \ \mathrm{d}y = \mathrm{area}(R) = 12.$$

□

6. Integrate the function given by $f(x, y) = xy^2$ over the region, R, defined by:

$$R = \{(x, y) \in \mathbb{R}^2 \mid x \geqslant 0, 0 \leqslant y \leqslant 4 - x^2\}.$$

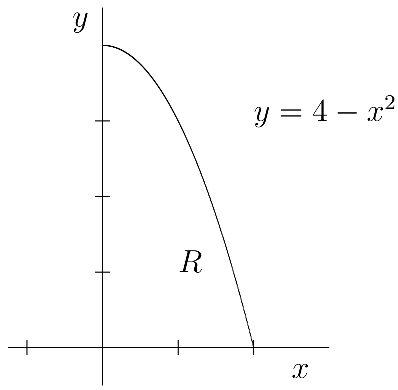

Figure 5: Sketch of Region R in Problem 8

***Solution**:* The region, R, is sketched in Figure 5. We evaluate the double integral, $\iint_R xy^2 \; \mathrm{d}x \, \mathrm{d}y$, as an iterated integral

$$
\begin{aligned}
\iint_R xy^2 \; \mathrm{d}x \, \mathrm{d}y &= \int_0^2 \int_0^{4-x^2} xy^2 \; \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^2 \int_0^{4-x^2} xy^2 \; \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^2 \frac{xy^3}{3}\bigg|_0^{4-x^2} \; \mathrm{d}x \\
&= \frac{1}{3}\int_0^2 x(4-x^2)^3 \; \mathrm{d}x.
\end{aligned}
$$

To evaluate the last integral, make the change of variables: $u = 4 - x^2$. We then have that $\mathrm{d}u = -2x \; \mathrm{d}x$ and

$$
\begin{aligned}
\iint_R xy^2 \; \mathrm{d}x \, \mathrm{d}y &= \int_0^2 \int_0^{4-x^2} xy^2 \; \mathrm{d}y \, \mathrm{d}x \\
&= -\frac{1}{6} \int_4^0 u^3 \; \mathrm{d}u \\
&= \frac{1}{6} \int_0^4 u^3 \; \mathrm{d}u.
\end{aligned}
$$

Thus,

$$\iint_R xy^2 \ \mathrm{d}x\ \mathrm{d}y \quad = \quad \frac{4^4}{24} = \frac{32}{3}.$$

□

7. Let R denote the region in the plane defined by inside of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1}$$

for $a > 0$ and $b > 0$.

(a) Evaluate the line integral $\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x$, where ∂R is the ellipse in (1) traversed in the positive sense.

Solution: A sketch of the ellipse is shown in Figure 6 for the case $a < b$.

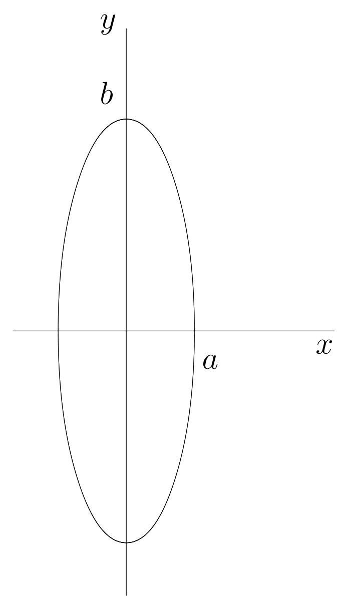

Figure 6: Sketch of ellipse

A parametrization of the ellipse is given by

$$x = a\cos t, \quad y = b\sin t, \quad \text{for} \ \ 0 \leqslant t \leqslant 2\pi.$$

We then have that $\mathrm{d}x = -a\sin t\,\mathrm{d}t$ and $\mathrm{d}y = b\cos t\,\mathrm{d}t$. Therefore

$$\begin{aligned}
\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x &= \int_0^{2\pi} [a\cos t\cdot b\cos t - b\sin t\cdot(-a\cos t)]\ \mathrm{d}t \\
&= \int_0^{2\pi} [ab\cos^2 t + ab\sin^2 t]\ \mathrm{d}t \\
&= ab\int_0^{2\pi} (\cos^2 t + ab\sin^2 t)\ \mathrm{d}t \\
&= ab\int_0^{2\pi}\ \mathrm{d}t \\
&= 2\pi ab.
\end{aligned}$$

□

(b) Use your result from part (a) and the divergence form of Green's theorem to come up with a formula for computing the area of the region enclosed by the ellipse in (1).

Solution: Let $F(x,y) = x\ \widehat{i} + y\ \widehat{j}$. Then,

$$\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x = \oint_{\partial R} F\cdot n\ \ \mathrm{d}s.$$

Thus, by Green's Theorem in divergence form,

$$\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x = \iint_R \mathrm{div}F\ \ \mathrm{d}x\ \mathrm{d}y,$$

where

$$\mathrm{div}F(x,y) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2.$$

Consequently,

$$\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x = 2\iint_R\ \mathrm{d}x\ \mathrm{d}y = 2\ \mathrm{area}(R).$$

It then follows that

$$\mathrm{area}(R) = \frac{1}{2}\oint_{\partial R} x\ \mathrm{d}y - y\ \mathrm{d}x.$$

Thus,

$$\mathrm{area}(R) = \pi ab,$$

by the result in part (a). □

8. Evaluate the double integral $\int_R e^{-x^2}\ \mathrm{d}x\ \mathrm{d}y$, where R is the region in the xy–plane sketched in Figure 7.

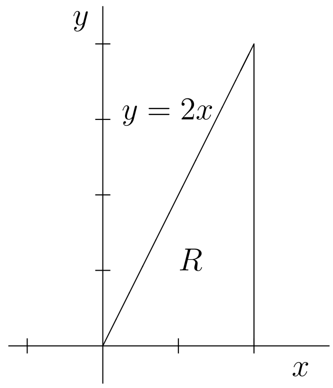

Figure 7: Sketch of Region R in Problem 8

***Solution**:* Compute

$$\begin{aligned} \iint_R e^{-x^2}\ \mathrm{d}x\ \mathrm{d}y &= \int_0^2 \int_0^{2x} e^{-x^2}\ \mathrm{d}y\ \mathrm{d}x \\ &= \int_0^2 2xe^{-x^2}\ \mathrm{d}x \\ &= \left[-e^{-x^2}\right]_0^2 \\ &= 1 - e^{-4}. \end{aligned}$$

□