Topics for Exam 3

1. Path Integrals
   1.1 $C^1$ curves and parametrizations
   1.2 Arclength
   1.3 Definition of the path integral

2. Line Integrals
   2.1 Definition of the line integral
   2.2 Piecewise $C^1$ curves.
   2.3 Simple, closed curves
   2.4 Flux across a closed curve in the plane

3. The Fundamental Theorem of Calculus
   3.1 Gradient fields
   3.2 Divergence and flux
   3.3 Green’s Theorem: Divergence form
   3.4 Double integrals

Relevant chapters and sections in the text: Section 7.4 on The Derivative, Section 7.6 on The Chain Rule, Section 3.1 on The Calculus of Curves, Section 5.2 on Line Integrals and Section 5.4 on Multiple Integrals.

Relevant chapters in the online class notes: Sections 4.5 and 4.6, Chapter 5, excluding sections 5.6, 5.7 and 5.9.

Important Concepts: $C^1$ curves, piecewise $C^1$ curves, simple curves, simple closed curves, parametrizations, arclength, path integral, line integral, flux, divergence and double integrals

Important Skills: Know how to evaluate the arclength of $C^1$ curves, know how to evaluate path integrals, know how to evaluate line integrals, know how to compute flux across a simple closed curve, know how to compute the divergence of a vector field, know how to evaluate double integrals, know how to apply the divergence form of Green’s theorem.
Some Formulas

1. **Tangent Line Approximation to a $C^1$ Path**

   The tangent line approximation to a $C^1$ path $\sigma: [a, b] \to \mathbb{R}^n$ at $\sigma(t_o)$, for some $t_o \in (a, b)$, is the straight line given by
   \[
   L(t) = \sigma(t_o) + (t - t_o)\sigma'(t_o) \quad \text{for all } t \in \mathbb{R}
   \]

2. **Arc Length**

   Let $\sigma: [a, b] \to \mathbb{R}^n$ be a $C^1$ parametrization of a curve $C$. The arc length of $C$ is given by
   \[
   \ell(C) = \int_a^b \|\sigma'(t)\| \, dt.
   \]

3. **Path Integral**

   Let $f: U \to \mathbb{R}$ be a continuous scalar field defined on some open subset of $\mathbb{R}^n$. Suppose there is a $C^1$ curve $C$ contained in $U$. Then the integral of $f$ over $C$ is given by
   \[
   \int_C f \, ds = \int_a^b f(\sigma(t))\|\sigma'(t)\| \, dt,
   \]
   for any $C^1$ parametrization, $\sigma: [a, b] \to \mathbb{R}^n$ of the curve $C$.

4. **Line Integral**

   Let $F: U \to \mathbb{R}^n$ denote a continuous vector field defined on some open subset, $U$, of $\mathbb{R}^n$. Suppose there is a $C^1$ curve, $C$, contained in $U$. Then, the line integral of $F$ over $C$ is given by
   \[
   \int_C F \cdot T \, ds = \int_a^b F(\sigma(t)) \cdot \sigma'(t) \, dt,
   \]
   for any $C^1$ parametrization, $\sigma: [a, b] \to \mathbb{R}^n$, of the curve $C$. Here $T$ denotes the tangent unit vector to the curve, and it is given by
   \[
   T(t) = \frac{1}{\|\sigma'(t)\|}\sigma'(t) \quad \text{for all } t \in (a, b).
   \]

   If $F = P \, \hat{i} + Q \, \hat{j} + R \, \hat{k}$, where $P$, $Q$, and $R$ are $C^1$ scalar fields defined on $U$,
   \[
   \int_C F \cdot T \, ds = \int_C P \, dx + Q \, dy + R \, dz.
   \]
   The expression $P \, dx + Q \, dy + R \, dz$ is called a differential 1–form.
5. Flux

Let \( F = P \hat{i} + Q \hat{j} \), where \( P \) and \( Q \) are continuous scalar fields defined on an open subset, \( U \), of \( \mathbb{R}^2 \). Suppose there is a \( C^1 \) simple closed curve \( C \) contained in \( U \). Then the flux of \( F \) across \( C \) is given by

\[
\oint_C F \cdot \hat{n} \, ds = \oint_C P \, dy - Q \, dx.
\]

Here, \( \hat{n} \) denotes a unit vector perpendicular to \( C \) and pointing to the outside of \( C \).

6. Green’s Theorem: Divergence Form

The Fundamental Theorem of Calculus in \( \mathbb{R}^2 \):

Let \( R \) denote a region in \( \mathbb{R}^2 \) bounded by a simple closed curve, \( \partial R \), made up of a finite number of \( C^1 \) paths traversed in the counterclockwise sense. Let \( P \) and \( Q \) denote two \( C^1 \) scalar fields defined on some open set containing \( R \) and its boundary, \( \partial R \). Then,

\[
\iint_R \text{div} F \, dx \, dy = \oint_{\partial R} F \cdot \hat{n} \, ds.
\]