Solutions to Assignment #11

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential(β), for $\beta > 0$.

   (a) Find a maximum likelihood estimator, $\widehat{\beta}$, for β.

   Solution: The likelihood function is

   $$L(\beta \mid x_1, x_2, \ldots, x_n) = f(x_1 \mid \beta) \cdot f(x_2 \mid \beta) \cdots f(x_n \mid \beta),$$

   where

   $$f(x \mid \beta) = \begin{cases} \dfrac{1}{\beta}\, e^{-x/\beta} & \text{if } x > 0; \\ \\ 0 & \text{otherwise.} \end{cases}$$

   It then follows that

   $$L(\beta \mid x_1, x_2, \ldots, x_n) = \frac{1}{\beta^n}\, e^{-y/\beta},$$

   where $y = \displaystyle\sum_{i=1}^{n} x_i$.

   In order to find the MLE for β, we maximize the funtion

   $$\ell(\beta) = \ln(L(\beta \mid x_1, x_2, \ldots, x_n)) = -\frac{y}{\beta} - n \ln \beta, \quad \text{for} \quad \beta > 0.$$

   Taking derivatives we obtain

   $$\ell'(\beta) = \frac{y}{\beta^2} - \frac{n}{\beta},$$

   and

   $$\ell''(\beta) = -\frac{2y}{\beta^3} + \frac{n}{\beta^2},$$

   for $\beta > 0$. Thus, $\widehat{\beta} = \dfrac{1}{n} y$ is a critical point with

   $$\ell''(\widehat{\beta}) = -\frac{n}{\widehat{\beta}^2} < 0.$$

   Hence, $\widehat{\beta} = \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} X_1 = \overline{X}_n$, the sample mean, is the MLE for β.

   □

(b) Find the likelihood ratio statistic for the test of $\text{H}_o\colon\ \beta = \beta_o$ versus the alternative $\text{H}_1\colon\ \beta \neq \beta_o$.

Solution: The likelihood ratio statistic in this case is

$$\begin{aligned}\Lambda(x_1, x_2, \ldots, x_n) &= \frac{L(\beta_o \mid x_1, x_2, \ldots, x_n)}{L(\widehat{\beta} \mid x_1, x_2, \ldots, x_n)} \\ &= \left(\frac{\widehat{\beta}}{\beta_o}\right)^n e^{-y/\beta_o + y/\widehat{\beta}} \\ &= \left(\frac{\widehat{\beta}}{\beta_o}\right)^n e^{-n\widehat{\beta}/\beta_o + n},\end{aligned}$$

since $\widehat{\beta} = \dfrac{y}{n}$, where $y = \displaystyle\sum_{i=1}^{n} x_i$.

We then have that

$$\Lambda(x_1, x_2, \ldots, x_n) = e^n \left(\frac{\widehat{\beta}}{\beta_o}\right)^n e^{-n\widehat{\beta}/\beta_o}. \tag{1}$$

□

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential(β), for $\beta > 0$, and H_o and H_1 be as in Problem 1.

(a) Show that the likelihood ratio statistic, $\Lambda(x_1, x_2, \ldots, x_n)$, found in part (b) of Problem 1 is of the form $e^n\ t^n e^{-nt}$, where $t = \widehat{\beta}/\beta_o$.

Solution: Substituting t for $\widehat{\beta}/\beta_o$ in equation (1) we obtain

$$\Lambda(x_1, x_2, \ldots, x_n) = e^n t^n e^{-nt} \quad \text{for} \quad t > 0.$$

□

(b) Let $g(t) = e^n\ t^n e^{-nt}$ for $t \geqslant 0$. Show that $g(t) \leqslant g(1) = 1$ for all $t \leqslant 0$, and sketch the graph of g.

Solution: Compute the derivatives of g to get

$$\begin{aligned}g'(t) &= ne^n t^{n-1} e^{-nt} - ne^n t^n e^{-nt} = ne^n t^{n-1}(1-t)e^{-nt}, \text{ and} \\ g''(t) &= n(n-1)e^n t^{n-2} e^{-nt} - 2n^2 e^n t^{n-1} e^{-nt} + n^2 e^n t^n e^{-nt}.\end{aligned}$$

Thus, for $n > 1$, g has two critical points, $t = 0$ and $t = 1$. Observe that, for $n > 2$, $g''(0) = 0$ and $g''(1) = -n < 0$. So that, g has a maximum at $t = 1$, which we wanted to show.
To sketch the graph of g, observe that for $n \geqslant 1$, $g(0)$, and, by L'Hospital's rule,

$$\lim_{t\to\infty} g(t) = e^n \lim_{t\to\infty} \frac{t^n}{e^{nt}} = 0.$$

A sketch of the graph of $g(t)$, for $n = 10$ is shown in Figure 1. $\square$

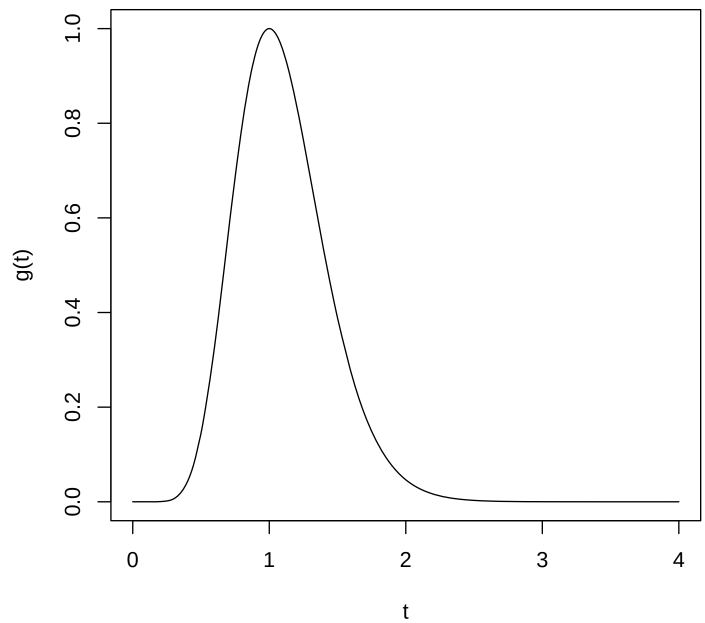

Figure 1: Sketch of graph of $g(t)$ for $n = 10$ and $0 \leqslant t \leqslant 4$

(c) Show that the rejection region R: $\Lambda(x_1, x_2, \ldots, x_n) \leqslant c$, for $0 < c < 1$, is equivalent to the region

$$\frac{1}{\beta_o}\overline{X}_n < c_1 \quad \text{or} \quad \frac{1}{\beta_o}\overline{X}_n > c_2,$$

for critical values c_1 and c_2 satisfying $0 < c_1 < 1/n < c_2$. Describe how you obtain c_1 and c_2 in terms of c.

Solution: By examining the graph of $\Lambda(t) = g(t)$ in Figure 1 we see that for $0 < c < 1$, the horizontal line at level c meets the graph of $g(t)$ at two points with t–coordinates at t_1 and t_2 with $0 < t_1 < 1 < t_2$; that is, $\Lambda(t_1) = \Lambda(t_2) = c$. Furthermore, since $g(t)$ is strictly increasing for $t < 1$, and strictly decreasing for $t > 1$, it follows that

$$\Lambda(t) \leqslant c \quad \text{iff} \quad t \leqslant t_1 \ \text{ or } \ t \geqslant t_2,$$

where

$$t = \frac{\widehat{\beta}}{\beta_o} = \frac{n\overline{X}_n}{\beta_o}.$$

In then follows that the LRT rejects H_o if

$$\frac{\overline{X}_n}{\beta_o} \leqslant \frac{t_1}{n} \quad \text{or} \quad \frac{\overline{X}_n}{\beta_o} \geqslant \frac{t_2}{n},$$

which was to be shown. □

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential(β), for $\beta > 0$, and H_o and H_1 be as in Problem 1.

   Define the statistic $Y = \dfrac{2}{\beta}\displaystyle\sum_{i=1}^{n} X_i$.

   (a) Assuming that H_o is true, give the distribution of the random variable Y.

   Solution: Since the X_is are iid random variables, the mgf of Y is given by

$$\begin{aligned} M_Y(t) &= \left[M_{X_1}\left(\frac{2t}{\beta}\right)\right]^n \\ &= \left[\frac{1}{1-\beta(2t/\beta)}\right]^n \\ &= \left[\frac{1}{1-2t}\right]^{2n/2} \end{aligned}$$

   for $t < \dfrac{1}{2}$, which is the mgf of a χ^2 distribution with $2n$ degrees of fredom. It then follows that

$$Y \sim \chi^2(2n),$$

   regardless of what β is. □

(b) Use the information gained in part (a) to come up with values of c_1 and c_2 such that the rejection region

$$R\colon \quad \frac{1}{\beta_o}\overline{X}_n < c_1 \quad \text{or} \quad \frac{1}{\beta_o}\overline{X}_n > c_2$$

yields a test with significance level α.

Solution: Observe that

$$\frac{1}{\beta_o}\overline{X}_n = \frac{1}{2n}Y$$

if $\beta = \beta_o$. Consequently,

$$\begin{aligned}\alpha &= \text{P}\left(\frac{1}{\beta_o}\overline{X}_n < c_1 \quad \text{or} \quad \frac{1}{\beta_o}\overline{X}_n > c_2\right) \\ &= \text{P}\left(\frac{1}{2n}Y < c_1 \quad \text{or} \quad \frac{1}{2n}Y > c_2\right),\end{aligned}$$

where $Y \sim \chi^2(2n)$.
Thus,

$$\begin{aligned}\alpha &= 1 - \text{P}\left(c_1 \leqslant \frac{1}{2n}Y \leqslant c_2\right) \\ &= 1 - \text{P}\left(2nc_1 \leqslant Y \leqslant 2nc_2\right) \\ &= 1 - \text{P}\left(2nc_1 < Y \leqslant 2nc_2\right) \\ &= 1 - \left(F_{_Y}(2nc_2) - F_{_Y}(2nc_1)\right),\end{aligned}$$

where $F_{_Y}$ denotes the cdf of $Y \sim \chi^2(2n)$. Thus, to get an LRT with significance level α, we need to have

$$F_{_Y}(2nc_2) - F_{_Y}(2nc_1) = 1 - \alpha.$$

we may accomplish this by setting

$$F_{_Y}(2nc_1) = \frac{\alpha}{2} \quad \text{and} \quad F_{_Y}(2nc_2) = 1 - \frac{\alpha}{2}.$$

we therefore get that

$$c_1 = \frac{1}{2n}F_{_Y}^{-1}(\alpha/2) \quad \text{and} \quad c_2 = \frac{1}{2n}F_{_Y}^{-1}(1 - (\alpha/2)). \qquad (2)$$

□

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential(β), for $\beta > 0$, and H_o and H_1 be as in Problem 1. Let Y denote the statistic defined in Problem 3.

   (a) If $\beta \neq \beta_o$, give the distribution of the test statistic Y.

   Answer: The answer obtained in part (a) of Problem 3 is independent of β. Hence, $Y \sim \chi^2(2n)$. □

   (b) Find an expression for the power function $\gamma(\beta)$ for the test for $\beta \neq \beta_o$.

   Solution: $\gamma(\beta)$ is the probability that the test will reject the null hypothesis when $\beta \neq \beta_o$. It then follows that

$$\begin{aligned}\gamma(\beta) &= \text{P}\left(\frac{1}{\beta_o}\overline{X}_n < c_1 \quad \text{or} \quad \frac{1}{\beta_o}\overline{X}_n > c_2\right)\\ &= \text{P}\left(\frac{1}{2n}\frac{\beta}{\beta_o}Y < c_1 \quad \text{or} \quad \frac{1}{2n}\frac{\beta}{\beta_o}Y > c_2\right),\end{aligned}$$

   where $Y \sim \chi^2(2n)$. Thus,

$$\begin{aligned}\gamma(\beta) &= 1 - \text{P}\left(2nc_1\frac{\beta_o}{\beta} \leqslant Y \leqslant 2nc_2\frac{\beta_o}{\beta}\right)\\ &= 1 - \text{P}\left(2nc_1\frac{\beta_o}{\beta} < Y \leqslant 2nc_2\frac{\beta_o}{\beta}\right)\\ &= 1 - \left[F_{_Y}\left(2nc_2\frac{\beta_o}{\beta}\right) - F_{_Y}\left(2nc_1\frac{\beta_o}{\beta}\right)\right]\end{aligned}$$

   □

   (c) Sketch the graph of $\gamma(\beta)$ for $\beta_o = 1$, $n = 10$ and $\alpha = 0.05$.

   Solution: Since $\alpha = 0.05$, we get from the formulas in (2) that

$$2nc_1 = F_{_Y}^{-1}(0.025) \quad \text{and} \quad 2nc_2 = F_{_Y}^{-1}(0.975),$$

   where $Y \sim \chi^2(20)$. Thus,

$$2nc_1 \approx 9.59 \quad \text{and} \quad 2nc_2 \approx 34.17.$$

   We then have that

$$\gamma(\beta) \approx 1 - \left[F_{_Y}\left(\frac{34.17}{\beta}\right) - F_{_Y}\left(\frac{9.59}{\beta}\right)\right].$$

   A sketch of the graph of this function for values of β between 0 and 4 is shown in Figure 2 on page 7. □

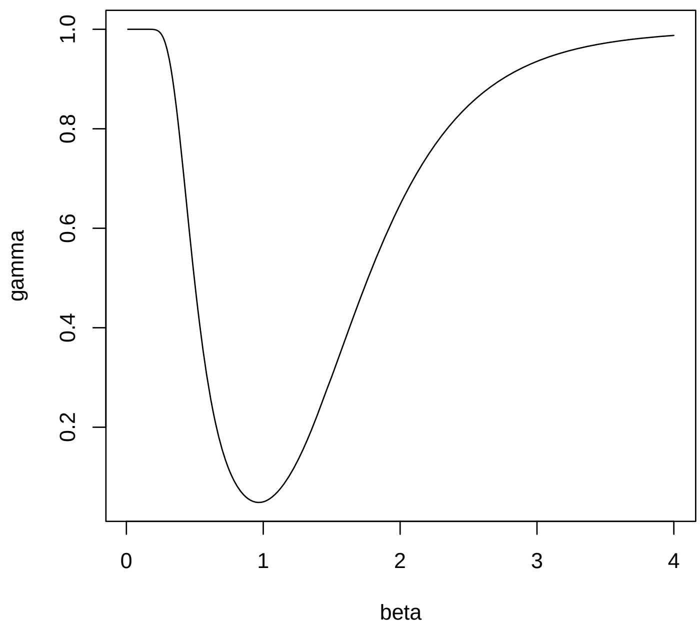

Figure 2: Sketch of graph of $\gamma(\beta)$ for $0 \leqslant \beta \leqslant 4$

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from an Poisson(λ), for $\lambda > 0$.

   (a) Find a maximum likelihood estimator, $\widehat{\lambda}$, for λ.

   Solution: The likelihood function in this case is

$$L(\lambda \mid x_1, x_2, \ldots, x_n) = \frac{\lambda^y}{x_1! x_2! \cdots x_n!}\ e^{-n\lambda},$$

where $y = \displaystyle\sum_{i=1}^{n} x_i$. To find the MLE for λ, we maximize the function

$$\ell(\lambda) = y \ln \lambda - n\lambda - \ln(x_1! x_2! \cdots x_n!).$$

Taking derivatives we obtain

$$\ell'(\lambda) = \frac{y}{\lambda} - n,$$

and

$$\ell''(\lambda) = -\frac{y}{\lambda^2}.$$

Thus, $\widehat{\lambda} = \dfrac{1}{n} y$ is a critical point of ℓ with $\ell''(\widehat{\lambda}) < 0$. It then follows that

$$\widehat{\lambda} = \overline{X}_n$$

is the MLE for λ.

□

   (b) Find the likelihood ratio statistic for the test of $\mathrm{H}_o\colon\ \lambda = \lambda_o$ versus the alternative $\mathrm{H}_1\colon\ \lambda \neq \lambda_o$.

   Solution: Compute

$$\begin{aligned}\Lambda(x_1, x_2, \ldots, x_n) &= \frac{L(\lambda_o \mid x_1, x_2, \ldots, x_n)}{L(\widehat{\lambda} \mid x_1, x_2, \ldots, x_n)} \\ &= \frac{\lambda_o^y\ e^{-n\lambda_o}}{\widehat{\lambda}^y\ e^{-n\widehat{\lambda}}} \\ &= \frac{1}{t^{n\lambda_o t}\ e^{n\lambda_o(1-t)}},\end{aligned}$$

where we have set $t = \dfrac{\widehat{\lambda}}{\lambda_o}$ and used $y = n\widehat{\lambda} = n\lambda_o t$. □

(c) Show that the likelihood ratio test of H_o versus H_1 is based on the test statistic $Y = \sum_{i=1}^{n} X_i$.

Solution: From our solution to part (b) of this problem, we see that, for a given sample size, n, and value of λ_o, the likelihood ratio statistic is a function of $t = \widehat{\lambda}/\lambda_o$, $\Lambda(t) = g(t)$, where

$$g(t) = \frac{1}{t^{n\lambda_o t}\ e^{n\lambda_o(1-t)}}, \quad \text{for } t > 0.$$

Note that $g(t)$ has a maximum value of 1 at $t = 1$. To see why this is so, let $h(t) = \ln(g(t))$ so that

$$h(t) = -n\lambda_o t\ \ln t - n\lambda_o(1-t),$$

and its derivatives are

$$h'(t) = -n\lambda_o\ \ln t$$

and

$$h''(t) = -\frac{n\lambda_o}{t}$$

for $t > 0$. It then follows that $t = 1$ is the only critical point for $h(t)$ in $(0, \infty)$ and $h''(1) < 0$. It then follows that $h(t)$ has a maximum at $t = 1$ and consequently $g(t)$ has a maximum at $t = 1$. Observe also that

$$\lim_{t \to 0^+} h(t) = -n\lambda_o$$

so that $g(0) = e^{-n\lambda_o}$, and

$$\lim_{t \to \infty} h(t) = -\infty,$$

so that

$$\lim_{t \to \infty} g(t) = 0.$$

We therefore get that the graph of $g(t)$ looks like the one sketched in Figure 3 on page 10, which is the sketch of the graph of $g(t)$ for $\lambda_o = 1$ and $n = 10$. We then see that for any c such that $0 < c < 1$, there exist two values of t, t_1 and t_2, such that $0 < t_1 < 1 < t_2$, and

$$g(t_1) = g(t_2) = c.$$

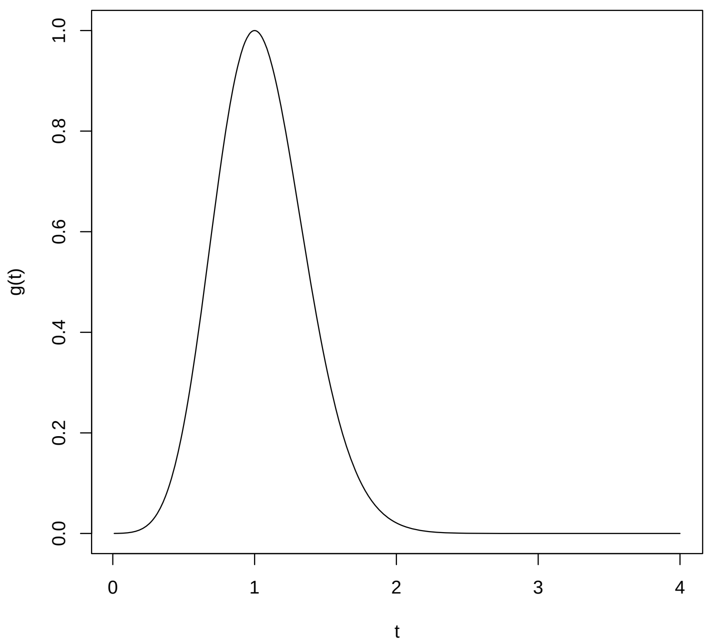

Figure 3: Sketch of graph of $g(t)$ for $0 < t < 4$, $\lambda_o = 1$, and $n = 10$

Furthermore,

$$g(t) \leqslant c \quad \text{for} \quad t \leqslant t_1 \quad \text{or} \quad t \geqslant t_2.$$

We then see that the LRT with rejection region

$$R\colon \quad \Lambda(x_1, x_2, \ldots, x_n) \leqslant c$$

is equivalent to the test that rejects H_o if

$$\frac{\widehat{\lambda}}{\lambda_o} \leqslant t_1 \quad \text{or} \quad \frac{\widehat{\lambda}}{\lambda_o} \geqslant t_2,$$

or

$$Y \leqslant n\lambda_o t_1 \quad \text{or} \quad Y \geqslant n\lambda_o t_2.$$

Hence, the LRT may be based on the test statistic $Y = \sum_{i=1}^{n} X_i$.

□

(d) Obtain the distribution of Y under the assumption that H_o is true.

Answer: The distribution of Y is Poisson($n\lambda_o$) if H_o is true. □

(e) For $\lambda_o = 2$ and $n = 5$, find the significance level of the the test that rejects H_o if either $Y \leqslant 4$ or $Y \geqslant 7$.

Solution: If $\lambda_o = 2$ and $n = 5$, then $Y \sim$ Poisson(10) if H_o is true. Then,

$$\begin{aligned}
\alpha &= \mathrm{P}(Y \leqslant 4 \text{ or } Y \geqslant 7) \\
&= \mathrm{P}(Y \leqslant 4) + \mathrm{P}Y \geqslant 7) \\
&= \mathrm{P}(Y \leqslant 4) + 1 - \mathrm{P}(Y \leqslant 6) \\
&= 1 - \left(\frac{10^5}{5!} + \frac{10^6}{6!}\right) e^{-10} \\
&\approx 0.9.
\end{aligned}$$

□