Solutions to Assignment #11

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential($\beta$), for $\beta > 0$.

   (a) Find a maximum likelihood estimator, $\hat{\beta}$, for $\beta$.

   **Solution:** The likelihood function is
   
   $$L(\beta \mid x_1, x_2, \ldots, x_n) = f(x_1 \mid \beta) \cdot f(x_2 \mid \beta) \cdots f(x_n \mid \beta),$$
   
   where
   
   $$f(x \mid \beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & \text{if } x > 0; \\ 0 & \text{otherwise}. \end{cases}$$
   
   It then follows that
   
   $$L(\beta \mid x_1, x_2, \ldots, x_n) = \frac{1}{\beta^n} e^{-y/\beta},$$
   
   where $y = \sum_{i=1}^{n} x_i$.

   In order to find the MLE for $\beta$, we maximize the function
   
   $$\ell(\beta) = \ln(L(\beta \mid x_1, x_2, \ldots, x_n)) = -\frac{y}{\beta} - n \ln \beta, \quad \text{for } \beta > 0.$$ 

   Taking derivatives we obtain
   
   $$\ell'(\beta) = \frac{y}{\beta^2} - \frac{n}{\beta},$$
   
   and
   
   $$\ell''(\beta) = -\frac{2y}{\beta^3} + \frac{n}{\beta^2},$$
   
   for $\beta > 0$. Thus, $\hat{\beta} = \frac{1}{n} y$ is a critical point with
   
   $$\ell''(\hat{\beta}) = -\frac{n}{\hat{\beta}^2} < 0.$$ 

   Hence, $\hat{\beta} = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}_n$, the sample mean, is the MLE for $\beta$. \qed
(b) Find the likelihood ratio statistic for the test of $H_0$: $\beta = \beta_o$ versus the alternative $H_1$: $\beta \neq \beta_o$.

**Solution:** The likelihood ratio statistic in this case is

$$\Lambda(x_1, x_2, \ldots, x_n) = \frac{L(\beta_o \mid x_1, x_2, \ldots, x_n)}{L(\hat{\beta} \mid x_1, x_2, \ldots, x_n)}$$

$$= \left(\frac{\hat{\beta}}{\beta_o}\right)^n e^{-y/\beta_o + y/\hat{\beta}}$$

$$= \left(\frac{\hat{\beta}}{\beta_o}\right)^n e^{-n\hat{\beta}/\beta_o + n},$$

since $\hat{\beta} = \frac{y}{n}$, where $y = \sum_{i=1}^{n} x_i$.

We then have that

$$\Lambda(x_1, x_2, \ldots, x_n) = e^n \left(\frac{\hat{\beta}}{\beta_o}\right)^n e^{-n\hat{\beta}/\beta_o}. \quad (1)$$

□

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential($\beta$), for $\beta > 0$, and $H_o$ and $H_1$ be as in Problem 1.

(a) Show that the likelihood ratio statistic, $\Lambda(x_1, x_2, \ldots, x_n)$, found in part (b) of Problem 1 is of the form $e^n t^n e^{-nt}$, where $t = \hat{\beta}/\beta_o$.

**Solution:** Substituting $t$ for $\hat{\beta}/\beta_o$ in equation (1) we obtain

$$\Lambda(x_1, x_2, \ldots, x_n) = e^n t^n e^{-nt} \quad \text{for } t > 0.$$  

□

(b) Let $g(t) = e^n t^n e^{-nt}$ for $t \geq 0$. Show that $g(t) \leq g(1) = 1$ for all $t \leq 0$, and sketch the graph of $g$.

**Solution:** Compute the derivatives of $g$ to get

$$g'(t) = ne^n t^{n-1} e^{-nt} - ne^n t^n e^{-nt} = ne^n t^{n-1}(1 - t)e^{-nt}, \quad \text{and}$$

$$g''(t) = n(n - 1)e^n t^{n-2} e^{-nt} - 2n^2 e^n t^{n-1} e^{-nt} + n^2 e^n t^n e^{-nt}.$$
Thus, for \( n > 1 \), \( g \) has two critical points, \( t = 0 \) and \( t = 1 \). Observe that, for \( n > 2 \), \( g''(0) = 0 \) and \( g''(1) = -n < 0 \). So that, \( g \) has a maximum at \( t = 1 \), which we wanted to show.

To sketch the graph of \( g \), observe that for \( n \geq 1 \), \( g(0) \), and, by L’Hospital’s rule,

\[
\lim_{t \to \infty} g(t) = e^n \lim_{t \to \infty} \frac{t^n}{e^{nt}} = 0.
\]

A sketch of the graph of \( g(t) \), for \( n = 10 \) is shown in Figure 1.

![Figure 1: Sketch of graph of \( g(t) \) for \( n = 10 \) and \( 0 \leq t \leq 4 \)](image)

(c) Show that the rejection region \( R: \Lambda(x_1, x_2, \ldots, x_n) \leq c \), for \( 0 < c < 1 \), is equivalent to the region

\[
\frac{1}{\theta_0} \overline{X}_n < c_1 \quad \text{or} \quad \frac{1}{\theta_0} \overline{X}_n > c_2,
\]

for critical values \( c_1 \) and \( c_2 \) satisfying \( 0 < c_1 < 1/n < c_2 \). Describe how you obtain \( c_1 \) and \( c_2 \) in terms of \( c \).
Solution: By examining the graph of \( \Lambda(t) = g(t) \) in Figure 1 we see that for \( 0 < c < 1 \), the horizontal line at level \( c \) meets the graph of \( g(t) \) at two points with \( t \)-coordinates at \( t_1 \) and \( t_2 \) with \( 0 < t_1 < 1 < t_2 \); that is, \( \Lambda(t_1) = \Lambda(t_2) = c \). Furthermore, since \( g(t) \) is strictly increasing for \( t < 1 \), and strictly decreasing for \( t > 1 \), it follows that

\[
\Lambda(t) \leq c \quad \text{iff} \quad t \leq t_1 \quad \text{or} \quad t \geq t_2,
\]

where

\[
t = \frac{\tilde{\beta}}{\beta_o} = \frac{nX_n}{\beta_o}.
\]

In then follows that the LRT rejects \( H_o \) if

\[
\frac{X_n}{\beta_o} < \frac{t_1}{n} \quad \text{or} \quad \frac{X_n}{\beta_o} > \frac{t_2}{n},
\]

which was to be shown. \( \square \)

3. Let \( X_1, X_2, \ldots, X_n \) be a random sample from an exponential(\( \beta \)), for \( \beta > 0 \), and \( H_o \) and \( H_1 \) be as in Problem 1.

Define the statistic \( Y = \frac{2}{\beta} \sum_{i=1}^{n} X_i \).

(a) Assuming that \( H_o \) is true, give the distribution of the random variable \( Y \).

Solution: Since the \( X_i \)s are iid random variables, the mgf of \( Y \) is given by

\[
M_Y(t) = \left[ M_{X_1} \left( \frac{2t}{\beta} \right) \right]^n
\]

\[
= \left[ \frac{1}{1 - \beta \left( \frac{2t}{\beta} \right) } \right]^n
\]

\[
= \left[ \frac{1}{1 - 2t} \right]^{2n/2}
\]

for \( t < \frac{1}{2} \), which is the mgf of a \( \chi^2 \) distribution with \( 2n \) degrees of freedom. It then follows that

\( Y \sim \chi^2(2n) \),

regardless of what \( \beta \) is. \( \square \)
(b) Use the information gained in part (a) to come up with values of $c_1$ and $c_2$ such that the rejection region

$$R: \frac{1}{\beta_o} X_n < c_1 \quad \text{or} \quad \frac{1}{\beta_o} X_n > c_2$$

yields a test with significance level $\alpha$.

**Solution:** Observe that

$$\frac{1}{\beta_o} X_n = \frac{1}{2n} Y$$

if $\beta = \beta_o$. Consequently,

$$\alpha = P \left( \frac{1}{\beta_o} X_n < c_1 \quad \text{or} \quad \frac{1}{\beta_o} X_n > c_2 \right)$$

$$= P \left( \frac{1}{2n} Y < c_1 \quad \text{or} \quad \frac{1}{2n} Y > c_2 \right),$$

where $Y \sim \chi^2(2n)$. Thus,

$$\alpha = 1 - P \left( c_1 \leq \frac{1}{2n} Y \leq c_2 \right)$$

$$= 1 - P \left( 2nc_1 \leq Y \leq 2nc_2 \right)$$

$$= 1 - P \left( 2nc_1 < Y \leq 2nc_2 \right)$$

$$= 1 - \left( F_Y(2nc_2) - F_Y(2nc_1) \right),$$

where $F_Y$ denotes the cdf of $Y \sim \chi^2(2n)$. Thus, to get an LRT with significance level $\alpha$, we need to have

$$F_Y(2nc_2) - F_Y(2nc_1) = 1 - \alpha.$$ 

we may accomplish this by setting

$$F_Y(2nc_1) = \frac{\alpha}{2} \quad \text{and} \quad F_Y(2nc_2) = 1 - \frac{\alpha}{2},$$

we therefore get that

$$c_1 = \frac{1}{2n} F_Y^{-1} \left( \frac{\alpha}{2} \right) \quad \text{and} \quad c_2 = \frac{1}{2n} F_Y^{-1} \left( 1 - \left( \frac{\alpha}{2} \right) \right). \quad (2)$$

$\square$
4. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential($\beta$), for $\beta > 0$, and $H_0$ and $H_1$ be as in Problem 1. Let $Y$ denote the statistic defined in Problem 3.

(a) If $\beta \neq \beta_o$, give the distribution of the test statistic $Y$.

**Answer:** The answer obtained in part (a) of Problem 3 is independent of $\beta$. Hence, $Y \sim \chi^2(2n)$. □

(b) Find an expression for the power function $\gamma(\beta)$ for the test for $\beta \neq \beta_o$.

**Solution:** $\gamma(\beta)$ is the probability that the test will reject the null hypothesis when $\beta \neq \beta_o$. It then follows that

$$\gamma(\beta) = P \left( \frac{1}{\beta_o} X_n < c_1 \text{ or } \frac{1}{\beta_o} X_n > c_2 \right)$$

$$= P \left( \frac{1}{2n \beta_o} Y < c_1 \text{ or } \frac{1}{2n \beta_o} Y > c_2 \right),$$

where $Y \sim \chi^2(2n)$. Thus,

$$\gamma(\beta) = 1 - P \left( 2nc_1 \frac{\beta_o}{\beta} \leq Y \leq 2nc_2 \frac{\beta_o}{\beta} \right)$$

$$= 1 - P \left( 2nc_1 \frac{\beta_o}{\beta} < Y \leq 2nc_2 \frac{\beta_o}{\beta} \right)$$

$$= 1 - \left[ F_Y \left( 2nc_2 \frac{\beta_o}{\beta} \right) - F_Y \left( 2nc_1 \frac{\beta_o}{\beta} \right) \right]$$

□

(c) Sketch the graph of $\gamma(\beta)$ for $\beta_o = 1, n = 10$ and $\alpha = 0.05$.

**Solution:** Since $\alpha = 0.05$, we get from the formulas in (2) that

$$2nc_1 = F_Y^{-1}(0.025) \quad \text{and} \quad 2nc_2 = F_Y^{-1}(0.975),$$

where $Y \sim \chi^2(20)$. Thus,

$$2nc_1 \approx 9.59 \quad \text{and} \quad 2nc_2 \approx 34.17.$$ 

We then have that

$$\gamma(\beta) \approx 1 - \left[ F_Y \left( \frac{34.17}{\beta} \right) - F_Y \left( \frac{9.59}{\beta} \right) \right].$$

A sketch of the graph of this function for values of $\beta$ between 0 and 4 is shown in Figure 2 on page 7. □
Figure 2: Sketch of graph of $\gamma(\beta)$ for $0 \leq \beta \leq 4$
5. Let \( X_1, X_2, \ldots, X_n \) be a random sample from an \( \text{Poisson}(\lambda) \), for \( \lambda > 0 \).

(a) Find a maximum likelihood estimator, \( \hat{\lambda} \), for \( \lambda \).

**Solution:** The likelihood function in this case is

\[
L(\lambda \mid x_1, x_2, \ldots, x_n) = \frac{\lambda^y e^{-n\lambda}}{x_1! x_2! \cdots x_n!},
\]

where \( y = \sum_{i=1}^{n} x_i \). To find the MLE for \( \lambda \), we maximize the function

\[
\ell(\lambda) = y \ln \lambda - n\lambda - \ln(x_1! x_2! \cdots x_n!).
\]

Taking derivatives we obtain

\[
\ell'(\lambda) = \frac{y}{\lambda} - n,
\]

and

\[
\ell''(\lambda) = -\frac{y}{\lambda^2}.
\]

Thus, \( \hat{\lambda} = \frac{1}{n} y \) is a critical point of \( \ell \) with \( \ell''(\hat{\lambda}) < 0 \). It then follows that

\[
\hat{\lambda} = \bar{X}_n
\]

is the MLE for \( \lambda \). \( \square \)

(b) Find the likelihood ratio statistic for the test of \( H_0: \lambda = \lambda_0 \) versus the alternative \( H_1: \lambda \neq \lambda_0 \).

**Solution:** Compute

\[
\Lambda(x_1, x_2, \ldots, x_n) = \frac{L(\lambda_0 \mid x_1, x_2, \ldots, x_n)}{L(\hat{\lambda} \mid x_1, x_2, \ldots, x_n)}
\]

\[
= \frac{\lambda_0^y e^{-n\lambda_0}}{\hat{\lambda}^y e^{-n\hat{\lambda}}}
\]

\[
= \frac{1}{t^{n\lambda_0} e^{-n\lambda_0(1-t)}},
\]

where we have set \( t = \frac{\hat{\lambda}}{\lambda_0} \) and used \( y = n\hat{\lambda} = n\lambda_0 t \). \( \square \)
(c) Show that the likelihood ratio test of $H_o$ versus $H_1$ is based on the test statistic $Y = \sum_{i=1}^{n} X_i$.

**Solution:** From our solution to part (b) of this problem, we see that, for a given sample size, $n$, and value of $\lambda_o$, the likelihood ratio statistic is a function of $t = \frac{\lambda}{\lambda_o}$, $\Lambda(t) = g(t)$, where

$$g(t) = \frac{1}{t^{n\lambda_o} e^{n\lambda_o(1-t)})}, \quad \text{for } t > 0.$$ 

Note that $g(t)$ has a maximum value of 1 at $t = 1$. To see why this is so, let $h(t) = \ln(g(t))$ so that

$$h(t) = -n\lambda_o t \ln t - n\lambda_o(1 - t),$$

and its derivatives are

$$h'(t) = -n\lambda_o \ln t$$

and

$$h''(t) = -\frac{n\lambda_o}{t}$$

for $t > 0$. It then follows that $t = 1$ is the only critical point for $h(t)$ in $(0, \infty)$ and $h''(1) < 0$. It then follows that $h(t)$ has a maximum at $t = 1$ and consequently $g(t)$ has a maximum at $t = 1$. Observe also that

$$\lim_{t \to 0^+} h(t) = -n\lambda_o$$

so that $g(0) = e^{-n\lambda_o}$, and

$$\lim_{t \to \infty} h(t) = -\infty,$$

so that

$$\lim_{t \to \infty} g(t) = 0.$$ 

We therefore get that the graph of $g(t)$ looks like the one sketched in Figure 3 on page 10, which is the sketch of the graph of $g(t)$ for $\lambda_o = 1$ and $n = 10$. We then see that for any $c$ such that $0 < c < 1$, there exist two values of $t$, $t_1$ and $t_2$, such that $0 < t_1 < 1 < t_2$, and

$$g(t_1) = g(t_2) = c.$$
Figure 3: Sketch of graph of $g(t)$ for $0 < t < 4$, $\lambda_o = 1$, and $n = 10$
Furthermore, 

\[ g(t) \leq c \text{ for } t \leq t_1 \text{ or } t \geq t_2. \]

We then see that the LRT with rejection region

\[ R: \quad \Lambda(x_1, x_2, \ldots, x_n) \leq c \]

is equivalent to the test that rejects \( H_0 \) if

\[ \frac{\hat{\lambda}}{\lambda_o} \leq t_1 \text{ or } \frac{\hat{\lambda}}{\lambda_o} \geq t_2, \]

or

\[ Y \leq n\lambda_o t_1 \text{ or } Y \geq n\lambda_o t_2. \]

Hence, the LRT may be based on the test statistic \( Y = \sum_{i=1}^n X_i \).

\( \square \)

(d) Obtain the distribution of \( Y \) under the assumption that \( H_0 \) is true.

**Answer:** The distribution of \( Y \) is Poisson\((n\lambda_o)\) if \( H_0 \) is true. \( \square \)

(e) For \( \lambda_o = 2 \) and \( n = 5 \), find the significance level of the test that rejects \( H_0 \) if either \( Y \leq 4 \) or \( Y \geq 7 \).

**Solution:** If \( \lambda_o = 2 \) and \( n = 5 \), then \( Y \sim \text{Poisson}(10) \) if \( H_0 \) is true. Then,

\[
\alpha = P(Y \leq 4 \text{ or } Y \geq 7)
\]

\[
= P(Y \leq 4) + P(Y \geq 7)
\]

\[
= P(Y \leq 4) + 1 - P(Y \leq 6)
\]

\[
= 1 - \left( \frac{10^5}{5!} + \frac{10^6}{6!} \right) e^{-10}
\]

\[
\approx 0.9.
\]

\( \square \)