Assignment #12

Due on Friday, November 13, 2009

Read Section 6.3 on Maximum Likelihood Tests, pp. 333–339, in Hogg, Craig and McKean.

Do the following problems

1. Suppose that you observe \( n \) iid Bernoulli(\( p \)) random variables, denoted by \( X_1, X_2, \ldots, X_n \). Find the LRT rejection region for the test of \( H_0: p \leq p_o \) versus \( H_1: p > p_o \) in terms of the test statistic \( Y = \sum_{i=1}^{n} X_i \).

2. Consider the likelihood ratio test for \( H_0: p = p_o \) versus \( H_1: p = p_1 \), where \( p_o \neq p_1 \), based on a random sample \( X_1, X_2, \ldots, X_n \) from a Bernoulli(\( p \)) distribution for \( 0 < p < 1 \). Show that, if \( p_1 > p_o \), then the likelihood ratio statistic for the test is a monotonically decreasing function of \( Y = \sum_{i=1}^{n} X_i \). Conclude, therefore, that if the test rejects \( H_o \) at the significance level \( \alpha \) for an observed value \( \hat{Y} \) of \( Y \), it will also rejects \( H_o \) at that level for \( Y > \hat{Y} \).

3. We wish to use an LRT to test the hypothesis \( H_o: \mu = \mu_o \) against the alternative \( H_1: \mu \neq \mu_o \) based on a random sample, \( X_1, X_2, \ldots, X_n \), from a normal(\( \mu, 1 \)) distribution.

   (a) Give the maximum likelihood estimator, \( \hat{\mu} \), for \( \mu \) based on the sample.

   (b) Give the likelihood ratio statistic for the test.

   (c) Express the LRT rejection region in terms of the sample mean \( \overline{X}_n \).

4. Let \( X_1, X_2, \ldots, X_n \) denote a random sample from a uniform(0, \( \theta \)) distribution for some parameter \( \theta > 0 \).

   (a) Give the likelihood function \( L(\theta \mid x_1, x_2, \ldots, x_n) \).

   (b) Give the maximum likelihood estimator for \( \theta \).
5. Let $R$ denote the rejection region for an LRT of $H_0: \theta = \theta_0$ versus $H_1: \theta = \theta_1$ based on a random sample, $X_1, X_2, \ldots, X_n$, from a continuous distribution with pdf $f(x | \theta)$. Let $L(\theta | x_1, x_2, \ldots, x_n)$ denote the likelihood function. Suppose the LRT has significance level $\alpha$.

(a) Explain why

$$\alpha = \int_R L(\theta_0 | x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \cdots dx_n.$$

(b) Explain why the power of the test is

$$\gamma(\theta_1) = \int_R L(\theta_1 | x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \cdots dx_n.$$

(c) Explain why

$$\alpha \leq c \gamma(\theta_1),$$

where $c$ is the critical value used in the definition of the rejection region, $R$, for the LRT.