Assignment #13

Due on Monday, November 16, 2009

Read Section 6.3 on Maximum Likelihood Tests, pp. 333–339, in Hogg, Craig and McKean.
Read Section 8.1 on Most Powerful Tests, pp. 419–427, in Hogg, Craig and McKean.
Read Section 8.2 on Uniformly Most Powerful Tests, pp. 429–435, in Hogg, Craig and McKean.

Do the following problems

1. Consider a test of the simple hypotheses

\[ H_o: \theta = \theta_o \quad \text{versus} \quad H_1: \theta = \theta_1 \]

based on a random sample from a distribution with pmf \( f(x \mid \theta) \), for \( x = 1, 2, \ldots, 7 \). The values of the likelihood function at \( \theta_o \) and \( \theta_1 \) are given in the table below.

\[
\begin{array}{c|cccccccc}
 x & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
L(\theta_o) & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.94 \\
L(\theta_1) & 0.06 & 0.05 & 0.04 & 0.03 & 0.02 & 0.01 & 0.79 \\
\end{array}
\]

Use the Neyman–Pearson Lemma to find the most powerful test for \( H_o \) versus \( H_1 \) with significance level \( \alpha = 0.04 \). Compute the probability of Type II error for this test.

2. Let \( X_1, X_2, \ldots, X_n \) be a random sample from a Poisson(\( \lambda \)) distribution.

(a) Find the most powerful test for testing

\[ H_o: \lambda = \lambda_o \quad \text{versus} \quad H_1: \lambda = \lambda_1, \]

for \( \lambda_1 > \lambda_o \).

(b) Show that the test found in part (a) is uniformly most powerful for testing

\[ H_o: \lambda = \lambda_o \quad \text{versus} \quad H_1: \lambda > \lambda_o. \]
3. Given a random sample, \( X_1, X_2, \ldots, X_n \), from a distribution with distribution function \( f(x \mid \theta) \). We say that a statistic \( T = T(X_1, X_2, \ldots, X_n) \) is **sufficient** for \( \theta \) is the joint distribution \( f(x_1, x_2, \ldots, x_n \mid \theta) \) can be written in the form

\[
f(x_1, x_2, \ldots, x_n \mid \theta) = g(T, \theta)h(x_1, x_2, \ldots, x_n),
\]

for some functions \( g: \mathbb{R}^2 \to \mathbb{R} \) and \( h: \mathbb{R}^n \to \mathbb{R} \).

Let \( X_1, X_2, \ldots, X_n \) be a random sample from a Poisson(\( \lambda \)) distribution. Find a sufficient statistic for \( \lambda \). Justify your answer based on the definition given above.

4. Suppose that \( X_1, X_2, \ldots, X_n \) forms a random sample from a distribution with distribution function \( f(x \mid \theta) \).

   (a) Show that if \( T \) is a sufficient statistic for \( \theta \), then the likelihood ratio statistic for the test of

   \[
   H_0: \theta = \theta_0 \quad \text{versus} \quad H_1: \theta = \theta_1
   \]

   is a function of \( T \).

   (b) Explain how knowledge of the distribution of \( T \) under \( H_0 \) may be used to choose a rejection region that yields the most powerful test at level \( \alpha \).

5. Derive a likelihood ratio test for

   \[
   H_0: \sigma^2 = \sigma^2_0 \quad \text{versus} \quad H_1: \sigma^2 \neq \sigma^2_0
   \]

   based on a sample from a normal(\( \mu, \sigma^2 \)) distribution.