Solutions to Assignment #13

1. Consider a test of the simple hypotheses

\[ H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1 \]

based on a random sample from a distribution with pmf \( f(x \mid \theta) \), for \( x = 1, 2, \ldots, 7 \). The values of the likelihood function at \( \theta_0 \) and \( \theta_1 \) are given in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L(\theta_0) )</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.94</td>
<td></td>
</tr>
<tr>
<td>( L(\theta_1) )</td>
<td>0.06</td>
<td>0.05</td>
<td>0.04</td>
<td>0.03</td>
<td>0.02</td>
<td>0.01</td>
<td>0.79</td>
</tr>
</tbody>
</table>

Use the Neyman–Pearson Lemma to find the most powerful test for \( H_0 \) versus \( H_1 \) with significance level \( \alpha = 0.04 \). Compute the probability of Type II error for this test.

**Solution:** Table 1 shows the values of the likelihood ratio statistic in the third row. Observe that if we let \( c = 1/3 \) and \( R \) the region defined by \( \Lambda \leq c \), then

\[ \alpha = P(\Lambda \leq 1/3 \mid \theta = \theta_0) = 0.04. \]

Thus, by the Neyman–Pearson Lemma, the test that rejects \( H_0 \) if

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</tr>
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<td>0.03</td>
<td>0.02</td>
<td>0.01</td>
<td>0.79</td>
</tr>
<tr>
<td>( L(\theta_0)/L(\theta_1) )</td>
<td>1/6</td>
<td>1/5</td>
<td>1/4</td>
<td>1/3</td>
<td>1/2</td>
<td>1</td>
<td>1.19</td>
</tr>
</tbody>
</table>

Table 1: Likelihood ratios

\[ \frac{L(\theta_0)}{L(\theta_1)} \leq \frac{1}{3} \]

is the most powerful test at significance level \( \alpha = 0.04 \).

The power of the test is

\[ \gamma(\theta_1) = P(\Lambda \leq 1/3 \mid \theta = \theta_1) = 0.18. \]

Thus, the probability of a Type II error is \( 1 - \gamma(\theta_1) = 82\% \).
2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson($\lambda$) distribution.

(a) Find the most powerful test for testing

$$H_0: \lambda = \lambda_o \quad \text{versus} \quad H_1: \lambda = \lambda_1,$$

for $\lambda_1 > \lambda_o$.

**Solution:** According to the Neyman–Pearson Lemma, the most powerful test is the LRT. To find the LRT rejection region, we first compute the likelihood ratio statistic

$$\Lambda(x_1, x_2, \ldots, x_n) = \frac{L(\lambda_o | x_1, x_2, \ldots, x_n)}{L(\lambda_1 | x_1, x_2, \ldots, x_n)},$$

where

$$L(\lambda | x_1, x_2, \ldots, x_n) = \frac{\lambda^y}{x_1!x_2!\cdots x_n!} e^{-n\lambda}, \quad (1)$$

for $y = \sum_{i=1}^{n} x_i$.

We then have that

$$\Lambda(x_1, x_2, \ldots, x_n) = \frac{\lambda_o^y e^{-n\lambda_o}}{\lambda_1^y e^{-n\lambda_1}} = \frac{e^{n\lambda_1}}{e^{n\lambda_o}} \left( \frac{\lambda_o}{\lambda_1} \right)^y = a^n r^y,$$

where we have set $a = e^{\lambda_1} / e^{\lambda_o}$ and $r = \lambda_o / \lambda_1$.

Since, $\lambda_1 > \lambda_o$, $a > 1$ and $r < 1$.

The rejection region of the LRT is

$$R: \quad \Lambda(x_1, x_2, \ldots, x_n) \leq c,$$

for some $c \in (0, 1)$ determined by the significance level of the test, or

$$a^n r^y \leq c, \quad (2)$$

where $y = \sum_{i=1}^{n} x_i$.

Taking the natural logarithm on both sides of the inequality in (2), we obtain

$$n \ln a + y \ln r \leq \ln c,$$

from which we get that

$$y \geq \frac{\ln c - n \ln a}{\ln r} \equiv b > 0.$$
Thus, the LRT rejects $H_0$ if

$$Y \geq b,$$

for some $b > 0$, where $Y = \sum_{i=1}^{n} X_i$. □

(b) Show that the test found in part (a) is uniformly most powerful for testing

$$H_o: \lambda = \lambda_o \text{ versus } H_1: \lambda > \lambda_o.$$

**Solution:** Fix $b$ in (3) so that $P(Y_0 > b) = \alpha$, where $Y_0 \sim \text{Poisson}(n\lambda_o)$. Note that this value of $b$ depends only on $\alpha$ and $\lambda_o$. Furthermore, by the result of part (a), the test that reject $H_o: \lambda = \lambda_o$ versus $H_o: \lambda = \lambda_1$, if

$$Y \geq b,$$

is the most powerful test at level $\alpha$ for every $\lambda_1 > \lambda_o$. It then follows that the test that rejects $H_o$ if

$$Y \geq b,$$

is the uniformly most powerful test of $H_o$ versus $H_1: \lambda > \lambda_o$. □

3. Given a random sample, $X_1, X_2, \ldots, X_n$, from a distribution with distribution function $f(x \mid \theta)$. We say that a statistic $T = T(X_1, X_2, \ldots, X_n)$ is **sufficient** for $\theta$ is the joint distribution $f(x_1, x_2, \ldots, x_n \mid \theta)$ can be written in the form

$$f(x_1, x_2, \ldots, x_n \mid \theta) = g(T, \theta)h(x_1, x_2, \ldots, x_n),$$

for some functions $g: \mathbb{R}^2 \to \mathbb{R}$ and $h: \mathbb{R}^n \to \mathbb{R}$.

Let $X_1, X_2, \ldots, X_n$ be a random sample from a $\text{Poisson}(\lambda)$ distribution. Find a sufficient statistic for $\lambda$. Justify your answer based on the definition given above.

**Solution:** According to (1) in the solution to part (a) of Problem 2 in this assignment, the likelihood function in this case is

$$L(\lambda \mid x_1, x_2, \ldots, x_n) = g(y, \lambda)h(x_1, x_2, \ldots, x_n),$$

where

$$g(y, \lambda) = \lambda^y e^{-\lambda}.$$
\[ h(x_1, x_2, \ldots, x_n) = \frac{1}{x_1!x_2! \cdots x_n!}, \]

and

\[ y = \sum_{i=1}^{n} x_i. \]

It then follows that \( Y = \sum_{i=1}^{n} X_i \) is a sufficient statistic for \( \lambda \). Observe that \( \bar{X}_n = \frac{1}{n} Y \) is also a sufficient statistic for \( \lambda \). \( \square \)

4. Suppose that \( X_1, X_2, \ldots, X_n \) forms a random sample from distribution with distribution function \( f(x \mid \theta) \).

(a) Show that if \( T \) is a sufficient statistic for \( \theta \), then the likelihood ratio statistic for the test of

\[ H_0: \theta = \theta_0 \quad \text{versus} \quad H_1: \theta = \theta_1 \]

is a function of \( T \).

\textbf{Solution:} In this case, the likelihood function is

\[ L(\theta \mid x_1, x_2, \ldots, x_n) = g(T, \theta)h(x_1, x_2, \ldots, x_n) \]

for all possible values of the parameter \( \theta \). It then follows that the likelihood ratio statistic is

\[ \Lambda(x_1, x_2, \ldots, x_n) = \frac{g(T, \theta_0)}{g(T, \theta_1)}, \]

which is a function of \( T \). \( \square \)

(b) Explain how knowledge of the distribution of \( T \) under \( H_0 \) may be used to choose a rejection region that yields the most powerful test at level \( \alpha \).

\textbf{Solution:} Knowing the distribution of \( T \), assuming that the null hypothesis is true, it is possible to find a value, \( c_\alpha \), for \( c \), such that

\[ P \left( \frac{g(T, \theta_0)}{g(T, \theta_1)} \leq c_\alpha \right) = \alpha. \]

The LRT rejection region is then given by

\[ R: \quad g(T, \theta_0) \leq c_\alpha g(T, \theta_1); \]

that is, if the value of \( T \) given by the sample falls in the region \( R \), the null hypothesis is rejected. \( \square \)
5. Derive a likelihood ratio test for

\[ H_0 : \sigma^2 = \sigma_o^2 \quad \text{versus} \quad H_1 : \sigma^2 \neq \sigma_o^2 \]

based on a sample from a normal(\(\mu, \sigma^2\)) distribution.

**Solution:** The likelihood ratio statistic is

\[ \Lambda(x_1, x_2, \ldots, x_n) = \sup_{\mu \in \mathbb{R}} \frac{L(\mu, \sigma_o | x_1, x_2, \ldots, x_n)}{L(\hat{\mu}, \hat{\sigma} | x_1, x_2, \ldots, x_n)}, \quad (4) \]

where \(\hat{\mu}\) and \(\hat{\sigma}^2\) are the MLEs for \(\mu\) and \(\sigma^2\), respectively; That is,

\[ \hat{\mu} = \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i, \]

and

\[ \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2. \]

When we maximize \(L(\mu, \sigma_o | x_1, x_2, \ldots, x_n)\) over \(\mu \in \mathbb{R}\) we obtain that \(\mu = \bar{x}\). It then follows from (4) that

\[ \Lambda(x_1, x_2, \ldots, x_n) = \frac{L(\hat{\mu}, \sigma_o | x_1, x_2, \ldots, x_n)}{L(\hat{\mu}, \hat{\sigma} | x_1, x_2, \ldots, x_n)}, \quad (5) \]

where

\[ L(\mu, \sigma | x_1, x_2, \ldots, x_n) = \frac{1}{(2\pi)^{n/2} \sigma^n e^{-\sum_{i=1}^{n}(x_i-\mu)^2/2\sigma^2}}, \]

for \(\mu \in \mathbb{R}\) and \(\sigma > 0\). We then have from (5) that

\[ \Lambda(x_1, x_2, \ldots, x_n) = \left( \frac{\hat{\sigma}}{\sigma_o} \right)^n \frac{e^{-\sum_{i=1}^{n}(x_i-\bar{x})^2/2\hat{\sigma}^2}}{e^{-\sum_{i=1}^{n}(x_i-\bar{x})^2/2\sigma_o^2}} \]

\[ = \left( \frac{\hat{\sigma}}{\sigma_o} \right)^n \frac{e^{-n\hat{\sigma}^2/2\sigma_o^2}}{e^{-n/2}} \]

\[ = e^{n/2} \frac{\hat{\sigma}^2}{\sigma_o^2} \]

where we have set \(t = \frac{\hat{\sigma}^2}{\sigma_o^2} \).
We then have that
\[ \Lambda(x_1, x_2, \ldots, x_n) = g(T) \]
where \( T \) is the statistic
\[ T = \frac{1}{n\sigma^2} \sum_{i=1}^{n} (X_i - \overline{X})^2, \tag{6} \]
and \( g(t) = e^{n/2} t^{n/2} e^{-nt/2} \). Observe that \( nT \) has a \( \chi^2(n - 1) \) distribution when the null hypothesis is true. Observe also that \( g(t) \) has a graph like the one sketched in Figure 1. It then follows that for any \( c \in (0, 1) \), there exists \( t_1 \) and \( t_2 \) such that \( 0 < t_1 < 1 < t_2 \) and
\[ g(t_1) = g(t_2) = c. \]
Furthermore,
\[ g(t) \leq c \quad \text{for} \quad t \leq t_1 \text{ or } t \geq t_2. \]
We then have that the LRT rejection region, 

\[ R: \Lambda(x_1, x_2, \ldots, x_n) \leq c, \]

can be expressed in terms of the statistic \( T \) in (6) as 

\[ R: \quad T \leq t_1 \text{ or } T \geq t_2. \]

The LRT rejection region can also be expressed in terms of the sample variance, \( S_n^2 \), as follows

\[ R: \quad S_n^2 \leq \frac{n-1}{n} t_1 \sigma_o^2 \text{ or } S_n^2 \geq \frac{n-1}{n} t_2 \sigma_o^2. \]

for \( 0 < t_1 < 1 < t_2 \), or, equivalently,

\[ R: \quad S_n^2 \leq \sigma_o^2 - b_1 \text{ or } S_n^2 \geq \sigma_o^2 + b_2, \]

for some positive values of \( b_1 \) and \( b_2 \) determined by the significance level of the test. \( \square \)