Solutions to Assignment #13

1. Consider a test of the simple hypotheses

$$\mathrm{H}_o\colon \theta = \theta_o \quad \text{versus} \quad \mathrm{H}_1\colon \theta = \theta_1$$

based on a random sample from a distribution with pmf $f(x \mid \theta)$, for $x = 1, 2, \ldots, 7$. The values of the likelihood function at θ_o and θ_1 are given in the table below.

x	1	2	3	4	5	6	7
$L(\theta_o)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$L(\theta_1)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79

Use the Neyman–Pearson Lemma to find the most powerful test for H_o versus H_1 with significance level $\alpha = 0.04$. Compute the probability of Type II error for this test.

***Solution**:* Table 1 shows the values of the likelihood ratio statistic in the third row. Observe that if we let $c = 1/3$ and R the region defined by $\Lambda \leqslant c$, then

$$\alpha = \mathrm{P}(\Lambda \leqslant 1/3 \mid \theta = \theta_o) = 0.04.$$

Thus, by the Neyman–Pearson Lemma, the test that rejects H_o if

x	1	2	3	4	5	6	7
$L(\theta_o)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$L(\theta_1)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79
$L(\theta_o)/L(\theta_1)$	1/6	1/5	1/4	1/3	1/2	1	1.19

Table 1: Likelihood ratios

$$\frac{L(\theta_o)}{L(\theta_1)} \leqslant \frac{1}{3}$$

is the most powerful test at significance level $\alpha = 0.04$.
The power of the test is

$$\gamma(\theta_1) = \mathrm{P}(\Lambda \leqslant 1/3 \mid \theta = \theta_1) = 0.18.$$

Thus, the probability of a Type II error is $1 - \gamma(\theta_1) = 82\%$. □

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson(λ) distribution.

   (a) Find the most powerful test for testing

$$\mathrm{H}_o\colon \lambda = \lambda_o \quad \text{versus} \quad \mathrm{H}_1\colon \lambda = \lambda_1,$$

   for $\lambda_1 > \lambda_o$.

   Solution: According to the Neyman–Pearson Lemma, the most powerful test is the LRT. To find the LRT rejection region, we first compute the likelihood ratio statistic

$$\Lambda(x_1, x_2, \ldots, x_n) = \frac{L(\lambda_o \mid x_1, x_2, \ldots, x_n)}{L(\lambda_1 \mid x_1, x_2, \ldots, x_n)},$$

   where

$$L(\lambda \mid x_1, x_2, \ldots, x_n) = \frac{\lambda^y}{x_1! x_2! \cdots x_n!}\, e^{-n\lambda}, \tag{1}$$

   for $y = \displaystyle\sum_{i=1}^{n} x_i$.
   We then have that

$$\Lambda(x_1, x_2, \ldots, x_n) = \frac{\lambda_o^y e^{-n\lambda_o}}{\lambda_1^y e^{-n\lambda_1}} = \frac{e^{n\lambda_1}}{e^{n\lambda_o}} \left(\frac{\lambda_o}{\lambda_1}\right)^y = a^n r^y,$$

   where we have set $a = e^{\lambda_1}/e^{\lambda_o}$ and $r = \lambda_o/\lambda_1$.
   Since, $\lambda_1 > \lambda_o$, $a > 1$ and $r < 1$.
   The rejection region of the LRT is

$$R\colon \quad \Lambda(x_1, x_2, \ldots, x_n) \leqslant c,$$

   for some $c \in (0, 1)$ determined by the significance level of the test, or

$$a^n r^y \leqslant c, \tag{2}$$

   where $y = \displaystyle\sum_{i=1}^{n} x_i$.
   Taking the natural logarithm on both sides of the inequality in (2), we obtain

$$n \ln a + y \ln r \leqslant \ln c,$$

   from which we get that

$$y \geqslant \frac{\ln c - n \ln a}{\ln r} \equiv b > 0.$$

Thus, the LRT rejects H_o if

$$Y \geqslant b, \tag{3}$$

for some $b > 0$, where $Y = \sum_{i=1}^{n} X_i$. □

(b) Show that the test found in part (a) is uniformly most powerful for testing

$$\mathrm{H}_o\colon \lambda = \lambda_o \quad \text{versus} \quad \mathrm{H}_1\colon \lambda > \lambda_o.$$

Solution: Fix b in (3) so that $\mathrm{P}(Y_o > b) = \alpha$, where $Y_o \sim \mathrm{Poisson}(n\lambda_o)$. Note that this value of b depends only on α and λ_o. Furthermore, by the result of part (a), the test that reject $\mathrm{H}_o\colon\ \lambda = \lambda_o$ versus $\mathrm{H}_o\colon\ \lambda = \lambda_1$, if

$$Y \geqslant b,$$

is the most powerful test at level α for every $\lambda_1 > \lambda_o$. It then follows that the test that rejects H_o if

$$Y \geqslant b,$$

is the uniformly most powerful test of H_o versus $\mathrm{H}_1\colon\ \lambda > \lambda_o$. □

3. Given a random sample, $X_1, X_2, \ldots, X_n$, from a distribution with distribution function $f(x \mid \theta)$. We say that a statistic $T = T(X_1, X_2, \ldots, X_n)$ is **sufficient** for θ is the joint distribution $f(x_1, x_2, \ldots, x_n \mid \theta)$ can be written in the form

$$f(x_1, x_2, \ldots, x_n \mid \theta) = g(T, \theta) h(x_1, x_2, \ldots, x_n),$$

for some functions $g\colon \mathbb{R}^2 \to \mathbb{R}$ and $h\colon \mathbb{R}^n \to \mathbb{R}$.

Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson(λ) distribution. Find a sufficient statistic for λ. Justify your answer based on the definition given above.

Solution: According to (1) int he solution to part (a) of Problem 2 in this assignment, the likelihood function in this case is

$$L(\lambda \mid x_1, x_2, \ldots, x_n) = g(y, \lambda) h(x_1, x_2, \ldots, x_n),$$

where

$$g(y, \lambda) = \lambda^y\ e^{-n\lambda},$$

$$h(x_1, x_2, \ldots, x_n) = \frac{1}{x_1!x_2!\cdots x_n!},$$

and

$$y = \sum_{i=1}^{n} x_i.$$

It then follows that $Y = \sum_{i=1}^{n} X_i$ is a sufficient statistic for λ. Observe that $\overline{X}_n = \frac{1}{n}Y$ is also a sufficient statistic for λ. □

4. Suppose that $X_1, X_2, \ldots, X_n$ forms a random sample from distribution with distribution function $f(x \mid \theta)$.

   (a) Show that if T is a sufficient statistic for θ, then the likelihood ratio statistic for the test of

   $$\text{H}_o\colon \theta = \theta_o \quad \text{versus} \quad \text{H}_1\colon \theta = \theta_1$$

   is a function of T.

   Solution: In this case, the likelihood function is

   $$L(\theta \mid x_1, x_2, \ldots, x_n) = g(T, \theta)h(x_1, x_2, \ldots, x_n)$$

   for all possible values of the parameter θ. It then follows that the likelihood ratio statistic is

   $$\Lambda(x_1, x_2, \ldots, x_n) = \frac{g(T, \theta_o)}{g(T, \theta_1)},$$

   which is a function of T. □

   (b) Explain how knowledge of the distribution of T under H_o may be used to choose a rejection region that yields the most powerful test at level α.

   Solution: Knowing the distribution of T, assuming that the null hypothesis is true, it is possible to find a value, c_α, for c, such that

   $$\text{P}\left(\frac{g(T, \theta_o)}{g(T, \theta_1)} \leqslant c_\alpha\right) = \alpha.$$

   The LRT rejection region is then given by

   $$R\colon \quad g(T, \theta_o) \leqslant c_\alpha g(T, \theta_1);$$

   that is, if the value of T given by the sample falls in the region R, the null hypothesis is rejected. □

5. Derive a likelihood ratio test for

$$\mathrm{H}_o\colon \sigma^2 = \sigma_o^2 \quad \text{versus} \quad \mathrm{H}_1\colon \sigma^2 \neq \sigma_o^2$$

based on a sample from a normal(μ, σ^2) distribution.

***Solution**:* The likelihood ratio statistic is

$$\Lambda(x_1, x_2, \ldots, x_n) = \frac{\sup\limits_{\mu\in\mathbb{R}} L(\mu, \sigma_o \mid x_1, x_2, \ldots, x_n)}{L(\widehat{\mu}, \widehat{\sigma} \mid x_1, x_2, \ldots, x_n)}, \tag{4}$$

where $\widehat{\mu}$ and $\widehat{\sigma}^2$ are the MLEs for μ and σ^2, respectively; That is,

$$\widehat{\mu} = \overline{x} = \frac{1}{n}\sum_{i=1}^n x_i,$$

and

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})^2.$$

When we maximize $L(\mu, \sigma_o \mid x_1, x_2, \ldots, x_n)$ over $\mu \in \mathbb{R}$ we obtain that $\mu = \overline{x}$. It then follows from (4) that

$$\Lambda(x_1, x_2, \ldots, x_n) = \frac{L(\widehat{\mu}, \sigma_o \mid x_1, x_2, \ldots, x_n)}{L(\widehat{\mu}, \widehat{\sigma} \mid x_1, x_2, \ldots, x_n)}, \tag{5}$$

where

$$L(\mu, \sigma \mid x_1, x_2, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}\ \sigma^n}\ e^{-\sum_{i=1}^n (x_i-\mu)^2/2\sigma^2},$$

for $\mu \in \mathbb{R}$ and $\sigma > 0$. We then have from (5) that

$$\begin{aligned}
\Lambda(x_1, x_2, \ldots, x_n) &= \left(\frac{\widehat{\sigma}}{\sigma_o}\right)^n \frac{e^{-\sum_{i=1}^n (x_i-\overline{x})^2/2\sigma_o^2}}{e^{-\sum_{i=1}^n (x_i-\overline{x})^2/2\widehat{\sigma}^2}} \\
&= \left(\frac{\widehat{\sigma}}{\sigma_o}\right)^n \frac{e^{-n\widehat{\sigma}^2/2\sigma_o^2}}{e^{-n/2}} \\
&= e^{n/2}\ t^{n/2}\ e^{-nt/2},
\end{aligned}$$

where we have set $t = \dfrac{\widehat{\sigma}^2}{\sigma_o^2}$.

We then have that

$$\Lambda(x_1, x_2, \ldots, x_n) = g(T)$$

where T is the statistic

$$T = \frac{1}{n\sigma_o^2} \sum_{i=1}^{n} (X_i - \overline{X})^2, \tag{6}$$

and $g(t) = e^{n/2}\ t^{n/2}\ e^{-nt/2}$. Observe that nT has a $\chi^2(n-1)$ distribution when the null hypothesis is true. Observe also that $g(t)$ has a graph like the one sketched in Figure 1. It then follows that for any

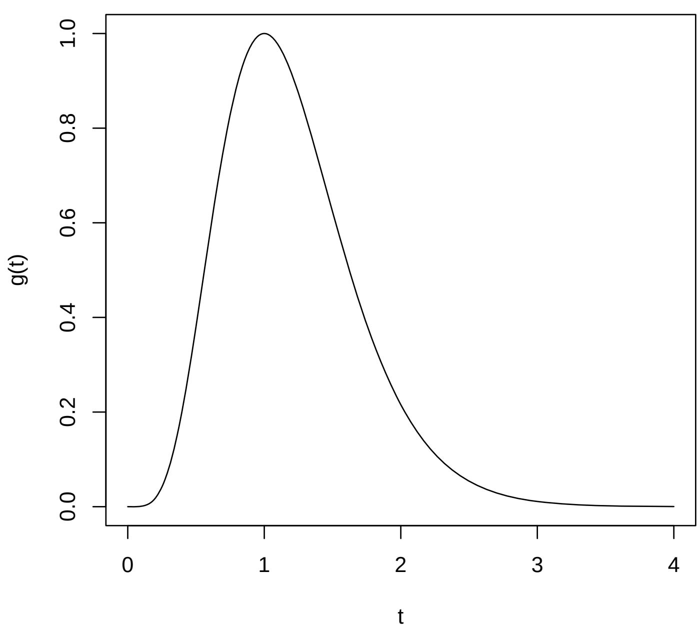

Figure 1: Sketch of graph of $g(t)$ for $n = 10$ and $0 \leqslant t \leqslant 4$

$c \in (0, 1)$, there exists t_1 and t_2 such that $0 < t_1 < 1 < t_2$ and

$$g(t_1) = g(t_1) = c.$$

Furthermore,

$$g(t) \leqslant c \quad \text{for} \quad t \leqslant t_1 \text{ or } t \geqslant t_2.$$

We then have that the LRT rejection region,

$$R: \quad \Lambda(x_, x_2, \ldots, x_n) \leqslant c,$$

can be expressed in terms of the statistic T in (6) as

$$R: \quad T \leqslant t_1 \text{ or } T \geqslant t_2.$$

The LRT rejection region can also be expressed in terms of the sample variance, S_n^2, as follows

$$R: \quad S_n^2 \leqslant \frac{n-1}{n} t_1 \sigma_o^2 \text{ or } S_n^2 \geqslant \frac{n-1}{n} t_2 \sigma_o^2,$$

for $0 < t_1 < 1 < t_2$, or, equivalently,

$$R: \quad S_n^2 \leqslant \sigma_o^2 - b_1 \text{ or } S_n^2 \geqslant \sigma_o^2 + b_2,$$

for some positive values of b_1 and b_2 determined by the significance level of the test. □