Assignment #14

Due on Monday, November 23, 2009

Read Section 6.2 on *Rao-Cramér lower bound and efficiency,* pp. 319–330, in Hogg, Craig and McKean.

Background and Definitions

Mean Squared Error, Bias and Efficiency

Let $X_1, X_2, \ldots, X_n$ denote a random sample from a distribution with distribution function $f(x \mid \theta)$, and let $W = W(X_1, X_2, \ldots, X_n)$ be an estimator for the parameter θ.

- **Mean Squared Error.** We define the **mean squared error** (MSE) of W to be the expected value of $(W-\theta)^2$. We denote the MSE of W by MSE(W) so that
$$\text{MSE}(W) = E_\theta\left[(W-\theta)^2\right]$$
- **Bias.** The bias of the estimator W is defined to be the quantity $E_\theta(W) - \theta$ and is denoted by $\text{bias}_\theta(W)$; thus,
$$\text{bias}_\theta(W) = E_\theta(W) - \theta.$$
- **MSE, variance and bias.**
$$\text{MSE}(W) = \text{var}_\theta(W) + [\text{bias}_\theta(W)]^2.$$
- **Efficiency.** If W and $\widetilde{W}$ are two unbiased estimators for θ the efficiency of $\widetilde{W}$ relative to W, denoted by $\text{eff}(\widetilde{W}, W)$, is defined to be
$$\text{eff}_\theta(\widetilde{W}, W) = \frac{\text{var}_\theta(\widetilde{W})}{\text{var}_\theta(W)}.$$

Do the following problems

1. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a Bernoulli(p) distribution with $0 < p < 1$. We have seen that $\widehat{p} = \dfrac{1}{n}\displaystyle\sum_{i=1}^n X_i$ is the MLE for p. Compute the mean squared error, MSE($\widehat{p}$), of $\widehat{p}$.

2. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a distribution with mean μ and variance σ^2.

   (a) For non–negative constants $a_1, a_2, \ldots, a_n$, define

   $$W = \sum_{i=1}^{n} a_i X_i. \tag{1}$$

   Prove that W is an unbiased estimator for μ if and only if $\displaystyle\sum_{i=1}^{n} a_i = 1$.

   (b) Out of all the unbiased estimators of μ of the form in (1), find the one which has the smallest possible variance. Calculate the variance of that estimator.

3. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a normal distribution with mean μ and variance σ^2.

   Compute the efficiency, $\text{eff}(\widehat{\sigma}^2, S_n^2)$, of $\widehat{\sigma}^2$, the MLE for σ^2, relative to the sample variance, S_n^2. What do you conclude?

4. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a Poisson distribution with parameter λ.

   (a) Show that the sample mean, $\overline{X}_n$, and the sample variance, S_n^2, are unbiased estimators of λ.

   (b) Compute the efficiency, $\text{eff}(\overline{X}_n, S_n^2)$, of $\overline{X}_n$ relative to S_n^2. What do you conclude?

5. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a uniform distribution over the interval $[0, \theta]$ for some parameter $\theta > 0$.

   We saw in Problem 4 of Assignment #12 that $W = \max\{X_1, X_2, \ldots, X_n\}$ is the MLE for θ. Determined whether W is unbiased or not.