Solutions to Assignment #14

1. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a Bernoulli(p) distribution with $0 < p < 1$. We have seen that $\widehat{p} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i$ is the MLE for p. Compute the mean squared error, MSE($\widehat{p}$), of $\widehat{p}$.

    Solution: Observe that $\widehat{p}$ is an unbiased estimator for p. It then follows that

    $$\text{MSE}(\widehat{p}) = \text{var}(\widehat{p}) = \frac{p(1-p)}{n}.$$

    □

2. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a distribution with mean μ and variance σ^2.

    (a) For non–negative constants $a_1, a_2, \ldots, a_n$, define

    $$W = \sum_{i=1}^{n} a_i X_i. \tag{1}$$

    Prove that W is an unbiased estimator for μ if and only if $\displaystyle\sum_{i=1}^{n} a_i = 1$.

    Solution: The estimator $W = \displaystyle\sum_{i=1}^{n} a_i X_i$ is an unbiased estimator for μ if and only if $E(W) = \mu$, if and only if,

    $$\sum_{i=1}^{n} a_i E(X_i) = \mu,$$

    if and only if

    $$\sum_{i=1}^{n} a_i \mu = \mu,$$

    if and only if

    $$\sum_{i=1}^{n} a_i = 1,$$

    which was to be shown. □

(b) Out of all the unbiased estimators of μ of the form in (1), find the one which has the smallest possible variance. Calculate the variance of that estimator.

Solution: For any estimator, W, of μ, which is of the form

$$W = \sum_{i=1}^{n} a_i X_i,$$

the variance is given by

$$\text{var}(W) = \sum_{i=1}^{n} a_i^2 \text{var}(X_i) = \sigma^2 \sum_{i=1}^{n} a_i^2,$$

since the X_is are independent. Thus, we need to minimize the function

$$f(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} a_i^2$$

subject to the constraint

$$g(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} a_i = 1.$$

We therefore use the method of lagrange multipliers; that is, we find λ and $a_1, a_2, \ldots, a_n$ such that

$$\nabla f(a_1, a_2, \ldots, a_n) = \lambda \nabla g(a_1, a_2, \ldots, a_n),$$

which leads to the equations

$$2a_i = \lambda \quad \text{for } i = 1, 2, \ldots, n,$$

or

$$a_i = \frac{\lambda}{2} \quad \text{for } i = 1, 2, \ldots, n.$$

Substituting these into the constraint equation,

$$g((a_1, a_2, \ldots, a_n) = 1,$$

yields

$$n\frac{\lambda}{2} = 1,$$

from which we get that

$$a_i = \frac{1}{n} \quad \text{for } i = 1, 2, \ldots, n.$$

Thus, the estimator

$$W = \sum_{i=i}^{n} \frac{1}{n} X_i = \overline{X}_n$$

provides a critical point for the variance over all estimators of the form

$$W = \sum_{i=1}^{n} a_i X_i,$$

with $\displaystyle\sum_{i=1}^{n} a_i = 1$. To see that $\overline{X}_n$ yields the smallest variance out of all estimators in the simplex defined by $\displaystyle\sum_{i=1}^{n} a_i = 1$, we compare $\text{var}(\overline{X}_n)$ with the variance at the corners of the simplex; namely,

$$\text{var}(X_i) = \sigma^2, \quad \text{for } i = 1, 2, \ldots, n.$$

Comparing these with

$$\text{var}(\overline{X}_n) = \frac{\sigma^2}{n},$$

we see that $\overline{X}_n$ has the smallest possible variance. □

3. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a normal distribution with mean μ and variance σ^2.

   Compute the efficiency, $\text{eff}(\widehat{\sigma}^2, S_n^2)$, of $\widehat{\sigma}^2$, the MLE for σ^2, relative to the sample variance, S_n^2. What do you conclude?

   ***Solution**:* Since $\widehat{\sigma}^2$ is not an unbiased estimator of σ^2, in this problem, it makes more sense to look at the ratio of the MSEs; that is, we consider the relative efficiency defined by

$$\text{eff}(\widehat{\sigma}^2, S_n^2) = \frac{\text{MSE}(\widehat{\sigma}^2)}{\text{MSE}(S_n^2)},$$

where

$$\text{MSE}(S_n^2) = \text{var}(S_n^2)$$

since S_n^n is an unbiased estimator of σ^2.

Using the fact that $\dfrac{n-1}{\sigma^2}S_n^2$ has a χ^2 with $n-1$ degrees of freedom we obtain that

$$\text{var}\left(\frac{n-1}{\sigma^2}S_n^2\right) = 2(n-1),$$

from which we get that

$$\text{var}\left(\frac{n-1}{\sigma^2}S_n^2\right) = 2(n-1),$$

or

$$\frac{(n-1)^2}{\sigma^4}\text{var}\left(S_n^2\right) = 2(n-1),$$

from which we get that

$$\text{var}\left(S_n^2\right) = \frac{2\sigma^4}{n-1}.$$

We therefore have that

$$\text{MSE}\left(S_n^2\right) = \frac{2\sigma^4}{n-1}.$$

Next, we compute the MSE of $\widehat{\sigma}^2$.

Observe first that $\widehat{\sigma}^2 = \dfrac{n-1}{n}S_n^2$, so that

$$E(\widehat{\sigma}^2) = \frac{n-1}{n}E(S_n^2) = \frac{n-1}{n}\sigma^2.$$

Thus $\widehat{\sigma}^2$ is biased with

$$\text{bias}(\widehat{\sigma}^2) = E(\widehat{\sigma}^2) - \sigma^2 = -\frac{\sigma^2}{n}.$$

Next, we compute the variance of $\widehat{\sigma}^2$:

$$\begin{aligned}
\text{var}(\widehat{\sigma}^2) &= \text{var}\left(\frac{n-1}{n}S_n^2\right) \\
&= \frac{(n-1)^2}{n^2}\text{var}\left(S_n^2\right) \\
&= \frac{(n-1)^2}{n^2}\cdot\frac{2\sigma^4}{n-1} \\
&= \frac{2(n-1)}{n^2}\cdot\sigma^4.
\end{aligned}$$

Thus,

$$\begin{aligned}
\text{MSE}(\widehat{\sigma}^2) &= \text{var}(\widehat{\sigma}^2) + \left(\text{bias}(\widehat{\sigma}^2)\right)^2 \\
&= \frac{2(n-1)}{n^2}\,\sigma^4 + \frac{1}{n^2}\,\sigma^2 \\
&= \frac{2n-1}{n^2}\,\sigma^4.
\end{aligned}$$

Thus,

$$\begin{aligned}
\text{eff}(\widehat{\sigma}^2, S_n^2) &= \frac{\text{MSE}(\widehat{\sigma}^2)}{\text{MSE}(S_n^2)} \\
&= \frac{\dfrac{2n-1}{n^2}\,\sigma^4}{\dfrac{2}{n-1}\sigma^4} \\
&= \frac{(2n-1)(n-1)}{2n^2} \\
&= 1 - \frac{1}{2n}\left(3 - \frac{1}{n}\right).
\end{aligned}$$

Hence,

$$\frac{\text{MSE}(\widehat{\sigma}^2)}{\text{MSE}(S_n^2)} < 1, \quad \text{for all} \;\; n,$$

which shows that the MLE $\widehat{\sigma}^2$ has a smaller mean square error that the unbiased estimator S_n^2. □

4. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a Poisson distribution with parameter λ.

   (a) Show that the sample mean, $\overline{X}_n$, and the sample variance, S_n^2, are unbiased estimators of λ.

   Solution: The mean and variance of a Poisson distribution with parameter λ are both equal to λ. It then follows that the sample mean, $\overline{X}_n$, and the sample variance, S_n^2, are both unbiased estimators of λ. □

   (b) Compute the efficiency, $\text{eff}(\overline{X}_n, S_n^2)$, of $\overline{X}_n$ relative to S_n^2. What do you conclude?

   Solution: We need to compute the variance of S_n^2. In order to do this we apply the formula

$$\text{var}(S_n^2) = \frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\mu_2^2\right), \tag{2}$$

where μ_2 denotes the second central moment, or variance, of the distribution and μ_4 is the fourth central moment. We therefore have from (2) that

$$\text{var}(S_n^2) = \frac{1}{n}\left(E\left[(X-\lambda)^4\right] - \frac{n-3}{n-1}\lambda^2\right), \tag{3}$$

where $X \sim \text{Poisson}(\lambda)$. Next, compute

$$\begin{aligned} E\left[(X-\lambda)^4\right] &= E\left[X^4 - 4\lambda X^3 + 6\lambda^2 X^2 - 4\lambda^3 X + \lambda^4\right] \\ &= E(X^4) - 4\lambda E(X^3) + 6\lambda^2 E(X^2) - 4\lambda^3 E(X) + \lambda^4, \end{aligned}$$

where we have used the linearity of the expectation operator. Thus,

$$E\left[(X-\lambda)^4\right] = E(X^4) - 4\lambda E(X^3) + 6\lambda^2 E(X^2) - 3\lambda^4. \tag{4}$$

Next, compute the moments $E(X^2)$, $E(X^3)$ and $E(X^4)$ by using the moment generating function

$$M_X(t) = e^{\lambda(e^t-1)}, \quad \text{for all } t \in \mathbb{R}.$$

Taking the first four derivatives we obtain

$$\begin{aligned} M_X'(t) &= \lambda e^t\, e^{\lambda(e^t-1)}, \\ M_X''(t) &= (\lambda^2 e^{2t} + \lambda e^t)\, e^{\lambda(e^t-1)}, \end{aligned}$$

$$M_X'''(t) = (\lambda^3 e^{3t} + 3\lambda^2 e^{2t} + \lambda e^t)\ e^{\lambda(e^t-1)},$$

and

$$M_X^{(4)}(t) = (\lambda^4 e^{4t} + 6\lambda^3 e^{3t} + 7\lambda^2 e^{2t} + \lambda e^t)\ e^{\lambda(e^t-1)}.$$

It then follows that the moments of X are

$$\begin{aligned} E(X^2) &= M_X''(0) = \lambda^2 + \lambda, \\ E(X^3) &= M_X'''(0) = \lambda^3 + 3\lambda^2 + \lambda, \\ E(X^4) &= M_X^{(4)}(0) = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda. \end{aligned}$$

Thus, using equation (4), we obtain that

$$E\left[(X-\lambda)^4\right] = 3\lambda^2 + \lambda.$$

We therefore get from (3) that

$$\begin{aligned} \mathrm{var}(S_n^2) &= \frac{1}{n}\left(3\lambda^2 + \lambda - \frac{n-3}{n-1}\lambda^2\right) \\ &= \frac{1}{n}\left(\frac{2n}{n-1}\lambda^2 + \lambda\right) \end{aligned}$$

The variance of $\overline{X}_n$ is

$$\mathrm{var}(\overline{X}_n) = \frac{\lambda}{n}.$$

We then have that

$$\mathrm{eff}(\overline{X}_n, S_n^2) = \frac{\mathrm{var}(\overline{X}_n)}{\mathrm{var}(S_n^2)} = \frac{n-1}{n-1+2n\lambda} < 1$$

for all n. Consequently, $\overline{X}_n$ is a more precise estimator of λ than the sample variance. □

5. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a uniform distribution over the interval $[0, \theta]$ for some parameter $\theta > 0$.

   We saw in Problem 4 of Assignment #12 that $W = \max\{X_1, X_2, \ldots, X_n\}$ is the MLE for θ. Determined whether W is unbiased or not.

***Solution**:* We need to compute the expected value of W:

$$E_\theta(W) = \int_{-\infty}^{\infty} w f_W(w \mid \theta) \; \mathrm{d}w, \qquad (5)$$

where $f_W(w \mid \theta)$ is the pdf of W. To determine the pdf of W, we first compute the cdf

$$\begin{aligned} F_W(w \mid \theta) &= \mathrm{P}(W \leqslant w) \\ &= \mathrm{P}(\max\{X_1, X_2, \ldots, X_n\} \leqslant w) \\ &= \mathrm{P}(X_1 \leqslant w, X_2 \leqslant w, \ldots, X_n \leqslant w) \\ &= \mathrm{P}(X_1 \leqslant w) \cdot \mathrm{P}(X_2 \leqslant w) \cdots \mathrm{P}(X_n \leqslant w), \end{aligned}$$

where we have used the independence of the X_is. Consequently, since the X_is are also identically distributed,

$$F_W(w \mid \theta) = [F_X(w)]^n.$$

it then follows that

$$f_W(w \mid \theta) = n[F_X(w)]^{n-1} f_X(w \mid \theta), \qquad (6)$$

where $f_X(w \mid \theta)$ is the pdf of $X \sim \text{uniform}[0, \theta]$; namely,

$$f_X(w \mid \theta) = \begin{cases} \dfrac{1}{\theta} & \text{if } \ 0 \leqslant w \leqslant \theta; \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$F_X(w \mid \theta) = \begin{cases} 0 & \text{if } w \leqslant 0; \\ \dfrac{w}{\theta} & \text{if } \ 0 < w \leqslant \theta; \\ 1 & \text{if } w > \theta. \end{cases}$$

Consequently, by equation (6),

$$f_W(w \mid \theta) = \begin{cases} \dfrac{n w^{n-1}}{\theta^n} & \text{if } \ 0 \leqslant w \leqslant \theta; \\ 0 & \text{otherwise,} \end{cases}$$

It then follows from 5)that

$$E_\theta(W) = \int_0^\theta \frac{nw^n}{\theta^n} \; \mathrm{d}\theta = \frac{n}{n+1}\,\theta.$$

We then see that $E_\theta(W) \neq \theta$, which shows that W is not an unbiased estimator of θ. □