Solutions to Assignment #15

1. Suppose that when the radius of a disc in the plane is measured, an error is made that has a normal(0, \( \sigma^2 \)) distribution. If \( n \) independent measurements are made, find an unbiased estimator for the area of the disc. Is this the best unbiased estimator for the area? Assume that \( \sigma^2 \) is known.

**Solution:** Let \( A \) denote the area of the disc and \( R_1, R_2, \ldots, R_n \) denote \( n \) independent measurements of the radius of the disc. By the information given in the problem, we may assume that

\[
R_i = \sqrt{\frac{A}{\pi}} + E_i, \quad \text{for } i = 1, 2, \ldots, n,
\]

where \( E_1, E_2, \ldots, E_n \) are iid normal(0, \( \sigma^2 \)) random variables. It then follows that \( R_1, R_2, \ldots, R_n \) are normal(\( \mu, \sigma^2 \)) random variables, where \( \mu = \sqrt{\frac{A}{\pi}} \). Then, the sample mean, \( \overline{R}_n \), is an unbiased estimator for \( \mu \). It is the best unbiased estimator for \( \mu \) in the sense that

\[
\text{var}(\overline{R}_n) = \frac{\sigma^2}{n}
\]

is the Cr\'amer–Rao lower bound. To see why this is the case, compute the information function

\[
I(\mu) = -E \left( \frac{\partial^2}{\partial \mu^2} \ln f(R \mid \mu, \sigma^2) \right),
\]

where

\[
f(r \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(r-\mu)^2}{2\sigma^2}}, \quad \text{for } r \in \mathbb{R},
\]

so that

\[
\ln f(r \mid \mu, \sigma^2) = -\frac{1}{2\sigma^2} (r - \mu)^2 - \ln(\sqrt{2\pi} \sigma).
\]

Thus,

\[
\frac{\partial}{\partial \mu} \ln f(r \mid \mu, \sigma^2) = \frac{1}{\sigma^2} (r - \mu),
\]

and

\[
\frac{\partial^2}{\partial \mu^2} \ln f(r \mid \mu, \sigma^2) = -\frac{1}{\sigma^2}.
\]
We then have that
\[ I(\mu) = -E\left(\frac{-1}{\sigma^2}\right) = \frac{1}{\sigma^2}. \]

Consequently, the Crâmer–Rao lower bound is
\[ \frac{1}{nI(\mu)} = \frac{\sigma^2}{n}, \]
which is attained by the variance of the sample mean, \( \overline{R}_n \). Hence, \( \overline{R}_n \) provides and unbiased estimator of \( \sqrt{\frac{A}{\pi}} \) with the lowest possible MSE. Thus, the formula \( \pi(\overline{R}_n)^2 \) provides the best unbiased estimator for the area, \( A \), of the disc. \( \square \)

2. Let \( X_1, X_2, \ldots, X_n \) be iid Bernoulli(\( p \)) random variables. Show that the MLE for \( p \) is an efficient estimator.

**Solution:** The MLE for \( p \) is the sample proportion \( \hat{p} = \overline{X}_n \). Thus, \( \hat{p} \) is also and unbiased estimator for \( p \). The variance of this estimator is
\[ \text{var}(\hat{p}) = \frac{p(1-p)}{n}. \]
To see that this in the Crâmer–Rao lower bound, we compute the information
\[ I(p) = -E\left(\frac{\partial^2}{\partial p^2} \ln f(x \mid p)\right), \]
where
\[ f(x \mid p) = p^x(1-p)^{1-x}, \quad \text{for } x = 0, 1. \]
Then,
\[ \ln f(x \mid p) = x \ln p + (1-x) \ln(1-p), \]
\[ \frac{\partial}{\partial p} \ln f(x \mid p) = \frac{x}{p} - \frac{(1-x)}{1-p}, \]
and
\[ \frac{\partial^2}{\partial p^2} \ln f(x \mid p) = -\frac{x}{p^2} - \frac{(1-x)}{(1-p)^2}. \]
Thus,

\[
I(p) = -E \left( \frac{X}{p^2} - \frac{(1 - X)}{(1 - p)^2} \right)
\]

\[= \frac{1}{p} + \frac{1}{1 - p}\]

\[= \frac{1}{p(1 - p)}.\]

Consequently, the Crámer–Rao lower bound is

\[
\frac{1}{nI(p)} = \frac{p(1 - p)}{n},
\]

which is attained by \(\operatorname{var}(\hat{p})\). Hence, \(\hat{p}\) is an efficient estimator of \(p\). \(\square\)

3. Let \(X_1, X_2, \ldots, X_n\) be iid exponential(\(\beta\)) random variables, and define

\(Y = \min\{X_1, X_2, \ldots, X_n\}\).

Find an unbiased estimator, \(W\), based only on \(Y\). Compute \(\operatorname{var}(W)\) and compare it to the variance of the sample mean, \(\overline{X}_n\). Which of \(W\) or \(\overline{X}_n\) is a more efficient estimator?

**Solution:** The common distribution function of the \(X_i\)s is

\[
f_X(x \mid \beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & \text{if } x > 0; \\ 0 & \text{otherwise}. \end{cases}
\]

Thus, the common cdf if

\[
F_X(x \mid \beta) = \begin{cases} 1 - e^{-x/\beta} & \text{if } x > 0; \\ 0 & \text{otherwise}. \end{cases}
\]

To find the distribution function of \(Y \min\{X_1, X_2, \ldots, X_n\}\), we first compute the cdf

\[
F_Y(y \mid \beta) = P(Y \leq y)
\]

\[= 1 - P(Y > y)\]

\[= 1 - P(X_1 > y, X_2 > y, \ldots, X_n > y)\]

\[= 1 - P(X_1 > y) \cdot P(X_2 > y) \cdots P(X_n > y),\]
where we have used the independence of the $X_i$s. Consequently, using the assumption that the $X_i$s are identically distributed, we obtain

$$F_Y(y | \beta) = 1 - [P(X > y)]^n$$

$$= 1 - [1 - P(X \leq y)]^n$$

$$= 1 - [1 - F_X(y | \beta)]^n.$$  

Thus, differentiating with respect to $y$ we have that

$$f_Y(y | \beta) = n[1 - F_X(y | \beta)]^{n-1}f_X(y),$$

where we have used the Chain Rule. It then follows that

$$f_Y(y | \beta) = \begin{cases} \frac{n}{\beta} e^{-ny/\beta} & \text{if } x > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, the expected value of $Y$ is

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y | \beta) \, dy$$

$$= \int_{0}^{\infty} y \frac{n}{\beta} e^{-ny/\beta} \, dy$$

$$= \frac{\beta}{n}.$$

We then have that $E(nY) = \beta$. Thus, if we set $W = nY$, we see that $W$ is an unbiased estimator of $\beta$.

Observe that $f_Y(y | \beta)$ is the pdf of an exponential($\beta/n$) distribution. It then follows that

$$\text{var}(Y) = \frac{\beta^2}{n^2}.$$  

Therefore,

$$\text{var}(W) = \text{var}(nY) = n^2 \text{var}(Y) = \beta^2.$$  

On the other hand, $\overline{X}_n$ is also an unbiased estimator of $\beta$. However,

$$\text{var}(\overline{X}_n) = \frac{\beta^2}{n} < \beta^2$$

for $n > 1$. We then have that $\text{var}(\overline{X}_n) < \text{var}(W)$ and therefore $\overline{X}_n$ is more efficient than $W$.  \[\square\]
4. Let \( X_1, X_2, \ldots, X_n \) be a random sample from a normal(\( \mu, \sigma^2 \)) distribution. Prove that the sample mean, \( \overline{X}_n \), is an efficient estimator of \( \mu \) for every known \( \sigma^2 > 0 \).

**Solution:** The information function is

\[
I(\mu) = -E\left( \frac{\partial^2}{\partial \mu^2} \ln f(X \mid \mu, \sigma^2) \right),
\]

where

\[
f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad \text{for } x \in \mathbb{R},
\]

so that

\[
\ln f(x \mid \mu, \sigma^2) = -\frac{1}{2\sigma^2} (x - \mu)^2 - \ln(\sqrt{2\pi} \sigma).
\]

Thus,

\[
\frac{\partial}{\partial \mu} \ln f(x \mid \mu, \sigma^2) = \frac{1}{\sigma^2} (x - \mu),
\]

and

\[
\frac{\partial^2}{\partial \mu^2} \ln f(x \mid \mu, \sigma^2) = -\frac{1}{\sigma^2}.
\]

We then have that

\[
I(\mu) = -E\left( -\frac{1}{\sigma^2} \right) = \frac{1}{\sigma^2}.
\]

Consequently, the Crámer–Rao lower bound is

\[
\frac{1}{nI(\mu)} = \frac{\sigma^2}{n},
\]

which is attained by the variance of the sample mean, \( \overline{X}_n \). Hence, \( \overline{X}_n \) is an efficient estimator of \( \mu \) for every known \( \sigma^2 > 0 \). \( \square \)

5. Let \( X_1, X_2, \ldots, X_n \) denote a random sample from a uniform distribution over the interval \([0, \theta]\) for some parameter \( \theta > 0 \).

Let \( Y = \max\{X_1, X_2, \ldots, X_n\} \) and define \( W = \frac{n + 1}{n} Y \). Compute the variance \( W \). Is \( W \) an efficient estimator of \( \theta \)?
Solution: We saw in Problem 5 of Assignment 14 that the pfd of $Y$ is given by

$$f_Y(y \mid \theta) = \begin{cases} \frac{ny^{n-1}}{\theta^n} & \text{if } 0 \leq y \leq \theta; \\ 0 & \text{otherwise}, \end{cases}$$

and that

$$E_Y(Y) = \frac{n}{n+1} \theta.$$ 

It then follows that

$$E(W) = E\left(\frac{n+1}{n}Y\right) = \frac{n+1}{n}E(Y) = \theta.$$ 

Hence, $W$ is an unbiased estimator of $\theta$.

To find the variance of $W$, we first compute the variance of $Y$:

$$\text{var}(Y) = E(Y^2) - [E(Y)]^2,$$

where

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y \mid \theta) \, dy$$

$$= \int_{0}^{\theta} y^2 \frac{ny^{n-1}}{\theta^n} \, dy$$

$$= \frac{n}{\theta^n} \int_{0}^{\theta} y^{n+1} \, dy$$

$$= \frac{n}{n+2} \theta^2.$$ 

Therefore,

$$\text{var}(Y) = \frac{n}{n+2} \theta^2 - \left[ \frac{n}{n+1} \theta \right]^2$$

$$= \frac{n}{n+2} \theta^2 - \frac{n^2}{(n+1)^2} \theta^2$$

$$= \frac{n}{(n+2)(n+1)^2} \theta^2.$$
We then have that
\[
\text{var}(W) = \text{var}\left(\frac{n+1}{n}Y\right) = \frac{(n+1)^2}{n^2}\text{var}(Y) = \frac{(n+1)^2}{n^2} \frac{n}{(n+2)(n+1)^2} \theta^2 = \frac{1}{n(n+2)} \theta^2.
\]

To see if \(W\) is an efficient estimator, we compute the information
\[
I(\theta) = \text{var}_\theta\left(\frac{\partial}{\partial \theta} \ln(f(X \mid \theta))\right) = E_\theta\left(\left[\frac{\partial}{\partial \theta} \ln(f(X \mid \theta))\right]^2\right),
\]
where
\[
f(x \mid \theta) = \begin{cases} 
\frac{1}{\theta} & \text{if } 0 \leq x \leq \theta; \\
0 & \text{otherwise},
\end{cases}
\]
so that
\[
\ln(f(x \mid \theta)) = \begin{cases} 
- \ln \theta & \text{if } 0 \leq x \leq \theta; \\
0 & \text{otherwise},
\end{cases}
\]
and
\[
\frac{\partial}{\partial \theta} \ln(f(x \mid \theta)) = \begin{cases} 
\frac{-1}{\theta} & \text{if } 0 \leq x \leq \theta; \\
0 & \text{otherwise}.
\end{cases}
\]
Thus,
\[
I(\theta) = \int_0^\theta \left(\frac{-1}{\theta}\right)^2 \frac{1}{\theta} \, dx = \frac{1}{\theta^2}.
\]
We then see that the Cramér–Rao lower bound is
\[
\frac{1}{nI(\theta)} = \frac{\theta^2}{n}.
\]
Note that this is larger than $\text{var}(W) = \frac{1}{n(n + 2)} \theta^2$. Thus, the Crámer–Rao inequality does not apply to this situation. To see why this is so, note that for any function $g$ of $x$,

\[
\frac{d}{d\theta} \int_{-\infty}^{\infty} g(x) f(x \mid \theta) \, dx = \frac{d}{d\theta} \int_{0}^{\theta} g(x) \frac{1}{\theta} \, dx
\]

\[
= \frac{d}{d\theta} \left( \frac{1}{\theta} \int_{0}^{\theta} g(x) \, dx \right)
\]

\[
= \frac{g(\theta)}{\theta} + \int_{0}^{\theta} g(x) \left( -\frac{1}{\theta^2} \right) \, dx,
\]

where we have used the Product Rule and the Fundamental Theorem of Calculus. On the other hand

\[
\int_{-\infty}^{\infty} \frac{\partial}{\partial \theta} (g(x) f(x \mid \theta)) \, dx = \frac{d}{d\theta} \int_{0}^{\theta} g(x) \frac{1}{\theta} \, dx
\]

\[
= \int_{0}^{\theta} g(x) \left( -\frac{1}{\theta^2} \right) \, dx.
\]

Thus, differentiation and integration can be interchanged if and only if

\[
\frac{g(\theta)}{\theta} = 0 \quad \text{for all } \theta.
\]

□