Solutions to Assignment #2

1. The reason that the function $M_X(t)$ is called the moment generating function for random variable $X$ is that the $n^{\text{th}}$ derivative of $M_X(t)$ at $t = 0$ is $E(X^n)$, the $n^{\text{th}}$ moment of the random variable $X$; that is,

$$M_X^{(n)}(0) = E(X^n) \quad \text{for } n = 1, 2, 3, \ldots \quad (1)$$

(a) Verify (1) for the case in which $X$ is continuous with pdf $f_X$. What assumptions do you need to make about the mgf in your derivation?

**Solution:** For the case of a continuous random variable, $X$, with pdf $f_X$, its mgf is given by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) \, dx,$$

for values of $t$ in some interval around 0. Assuming that

$$\int_{-\infty}^{\infty} |x|^n e^{tx} f_X(x) \, dx < \infty$$

for all $t$ in some interval around 0, since the functions

$$(x, t) \mapsto x^n e^{tx}$$

are continuous, it follows by differentiating under the integral sign with respect to $t$ that

$$M_X^{(n)}(t) = \int_{-\infty}^{\infty} x^n e^{tx} f_X(x) \, dx \quad \text{for all } n = 1, 2, 3, \ldots$$

and for $t$ in an interval around 0. Consequently,

$$M_X^{(n)}(0) = \int_{-\infty}^{\infty} x^n f_X(x) \, dx = E(X^n) \quad \text{for all } n = 1, 2, 3, \ldots$$

which was to be shown. \hfill \Box

(b) Show that if the mgf of $X$ exists on some interval around 0, then

$$\text{var}(X) = M_X''(0) - [M_X'(0)]^2$$
**Solution:** For this problem, we also need to assume that the mgf of $X$ is twice differentiable at 0. Then,

\[
\text{var}(X) = E[(X - \mu_X)^2]
\]

\[= E(X^2 - 2\mu_X X + \mu_X^2),\]

where $\mu_X = E(X)$. Thus, by the linearity of the expectation operator,

\[
\text{var}(X) = E(X^2) - 2\mu_X E(X) + \mu_X^2 E(1)
\]

\[= E(X^2) - 2\mu_X E(X) + \mu_X^2 E(1)
\]

\[= E(X^2) - 2\mu_X \mu_X + \mu_X^2
\]

\[= E(X^2) - \mu_X^2
\]

\[= E(X^2) - [E(X)]^2
\]

\[= M''_X(0) - [M'_X(0)]^2,
\]

which was to be shown. \[\square\]

2. Let $\lambda > 0$. A random variable $X$ is said to follow a Poisson($\lambda$) distribution if $X$ takes the values 0, 1, 2, 3, ... and the pmf of $X$ is given by

\[p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda} \text{ for all } k = 0, 1, 2, 3, \ldots\]

Compute the mgf of a Poisson($\lambda$) random variable, $X$. For which values of $t$ is the mgf defined?

**Solution:** Compute

\[M_X(t) = E(e^{tX})
\]

\[= \sum_{k=0}^{\infty} e^{tk} p_X(k)
\]

\[= \sum_{k=0}^{\infty} (e^t)^k \frac{\lambda^k}{k!} e^{-\lambda}.\]
It then follow that the mgf of $X \sim \text{Poisson}(\lambda)$ is

$$M_X(t) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$= e^{\lambda(e^t-1)},$$

for all $t \in \mathbb{R}$. □

3. Use the result of Problem 2 to compute the mean and variance of a Poisson($\lambda$) distribution. What do you discover?

**Solution:** Differentiating the mgf of $X$ obtained in Problem 2 with respect to $t$, we get

$$M'_X(t) = \lambda e^t e^{\lambda(e^t-1)},$$

and

$$M''_X(t) = \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)}.$$  

We then get that the expected value of $X$ is

$$E(X) = M'_X(0) = \lambda,$$

and the second moment of $X$ is

$$E(X^2) = M''_X(0) = \lambda + \lambda^2.$$

Consequently, by the result in part (b) of Problem 1, the variance of $X$ is

$$\text{var}(X) = E(X^2) - \lambda^2 = \lambda.$$

Thus, the expected value and variance of a Poisson random variable are the same. □

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson($\lambda$) distribution. Define $Y_n = X_1 + X_2 + \cdots + X_n$. Give the sampling distribution for $Y_n$. What do you discover?
**Solution:** Compute the mgf of $Y_n$, $M_{Y_n}(t) = E(e^{tY_n})$, to get that

$$M_{Y_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t),$$

where we have used the independence assumption. Thus, since the random variables $X_1, X_2, \ldots, X_n$ are identically distributed, it follows from the result of Problem 2 that

$$M_{Y_n}(t) = \left(e^{\lambda(t-1)}\right)^n = e^{n\lambda(e^t-1)},$$

which is the mgf of a Poisson($n\lambda$) random variable. It follows that $Y_n$ has a Poisson distribution with parameter $n\lambda$. □

5. Let $X_1, X_2, X_3 \ldots$ be a sequence of random variable satisfying $X_n \sim \text{binomial}(n, p)$ for all $n$. Assume also that $np = \lambda$, where $\lambda$ is a fixed parameter.

Compute $M_{X_n}(t)$ for all $n$ and determine the limit

$$\lim_{n \to \infty} M_{X_n}(t).$$

What do you discover?

*Hint:* Observe that $p = \frac{\lambda}{n} \to 0$ as $n \to \infty$ since $\lambda$ is assumed to be fixed.

**Solution:** Compute

$$M_{X_n}(t) = (1 - p + p e^t)^n = \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n} e^t\right)^n,$$

or

$$M_{X_n}(t) = \left(1 + \frac{\lambda(e^t - 1)}{n}\right)^n.$$

It then follows that

$$\lim_{n \to \infty} M_{X_n}(t) = \lim_{n \to \infty} \left(1 + \frac{\lambda(e^t - 1)}{n}\right)^n = e^{\lambda(e^t-1)},$$

which is the mgf of a Poisson($\lambda$) random variable. Note that we have used the definition of $e^u$ as

$$e^u = \lim_{n \to \infty} \left(1 + \frac{u}{n}\right)^n \quad \text{for all } u \in \mathbb{R}.\quad \square$$