Solutions to Assignment #3

1. Let X and Y be independent continuous random variables with pdfs f_X and f_Y, respectively. Let $W = X + Y$ and show that the pdf for W is given by

$$f_W(w) = \int_{-\infty}^{+\infty} f_X(u) f_Y(w-u)\, \mathrm{d}u \tag{1}$$

for all $w \in \mathbb{R}$. This is known as the *convolution* of f_X and f_Y.

Suggestion: To evaluate the double integral defining $P(X+Y \leq z)$, make the change of variables $u = x$ and $v = x + y$. Observe that with this change of variables, the region of integration in the uv–plane becomes:

$$\{(u,v) \in \mathbb{R}^2 \mid -\infty < u < \infty, -\infty < v < z\}.$$

Refer to pages 86 and 87 in the text on how to perform a change of variables for a double integral.

***Solution**:* We first compute the cdf

$$F_W(w) = P(W \leqslant w) \quad \text{for } w \in \mathbb{R},$$

where

$$\begin{aligned} P(W \leqslant w) &= P(X+Y \leqslant w) \\ &= \iint_{\{x+y\leqslant w\}} f_{(X,Y)}(x,y)\, \mathrm{d}x\, \mathrm{d}y. \end{aligned}$$

Since X and Y are independent, the joint pdf of X and Y is given by

$$f_{(X,Y)}(x,y) = f_X(x) \cdot f_Y(y).$$

We then have that

$$F_W(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{w-x} f_X(x) \cdot f_Y(y)\, \mathrm{d}y\, \mathrm{d}x,$$

see Figure 1.

Next, make the change of variables: $u = x$, $v = x + y$ to get that

$$F_W(w) = \int_{-\infty}^{w} \int_{-\infty}^{\infty} f_X(u) \cdot f_Y(v-u) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \mathrm{d}u\, \mathrm{d}v,$$

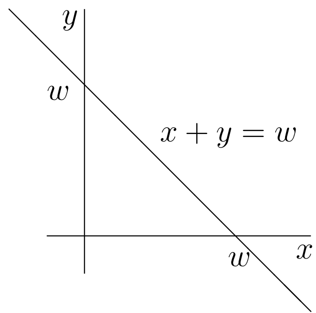

Figure 1: $\{x + y \leqslant w\}$

where the Jacobian of the change of variables is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = 1.$$

Consequently,

$$F_W(w) = \int_{-\infty}^{w}\int_{-\infty}^{\infty} f_X(u)\cdot f_Y(v-u)\ \mathrm{d}u\ \mathrm{d}v,$$

Next, differentiate with respect to w to obtain the pdf

$$f_W(w) = \int_{-\infty}^{\infty} f_X(u)\cdot f_Y(w-u)\ \mathrm{d}u,$$

where we have applied the Fundamental Theorem of Calculus, which is the convolution formula in (1). □

2. Let $X \sim$ exponential(2) and $Y \sim \chi^2(1)$ be independent random variables. Define $W = X + Y$. Use the convolution formula in (1) to find the pdf of W.

***Solution**:* Since X and Y are independent, f_W is the convolution of f_X and f_Y:

$$\begin{aligned} f_W(w) &= f_X * f_Y(w) \\ &= \int_{-\infty}^{\infty} f_X(u) f_Y(w-u)\ \mathrm{d}u, \end{aligned}$$

where

$$f_X(x) = \begin{cases} \dfrac{1}{2}e^{-x/2} & \text{if } x > 0; \\ \\ 0 & \text{otherwise;} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi}}\dfrac{1}{\sqrt{y}}\ e^{-y/2} & \text{if } y > 0; \\ \\ 0 & \text{otherwise.} \end{cases}$$

It then follows that, for $w > 0$,

$$\begin{aligned} f_W(w) &= \int_0^\infty \frac{1}{2}e^{-u/2} f_Y(w-u)\ \mathrm{d}u \\ &= \int_0^w \frac{1}{2}e^{-u/2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{w-u}}\ e^{-(w-u)/2}\ \mathrm{d}u \\ &= \frac{e^{-w/2}}{2\sqrt{2\pi}}\int_0^w \frac{1}{\sqrt{w-u}}\ \mathrm{d}u. \end{aligned}$$

Making the change of variables $t = u/w$, we get that $u = wt$ and $\mathrm{d}u = w\ \mathrm{d}t$, so that

$$\begin{aligned} f_W(w) &= \frac{e^{-w/2}}{2\sqrt{2\pi}}\int_0^1 \frac{1}{\sqrt{w-wt}}\ w\ \mathrm{d}t \\ &= \frac{\sqrt{w}\ e^{-w/2}}{2\sqrt{2\pi}}\int_0^1 \frac{1}{\sqrt{1-t}}\ \mathrm{d}t \\ &= \frac{\sqrt{w}\ e^{-w/2}}{\sqrt{2\pi}}\left[-\sqrt{1-t}\right]_0^1 \\ &= \frac{1}{\sqrt{2\pi}}\ \sqrt{w}\ e^{-w/2}, \end{aligned}$$

for $w > 0$. It then follows that

$$f_W(w) = \begin{cases} \dfrac{1}{\sqrt{2\pi}}\ \sqrt{w}\ e^{-w/2} & \text{if } w > 0; \\ \\ 0 & \text{otherwise.} \end{cases}$$

This is the pdf for a $\chi^2(3)$ random variable. $\square$

3. We use the notation $f_X * f_Y$ to denote the convolution of the two pdfs f_X and f_Y as defined in (1); that is,

$$f_X * f_Y(w) = \int_{-\infty}^{+\infty} f_X(u) f_Y(w-u)\, \mathrm{d}u \quad \text{for all } w \in \mathbb{R}.$$

Verify that convolution is a symmetric operation; that is,

$$f_X * f_Y = f_Y * f_X.$$

***Solution**:* Apply the definition of convolution to $f_Y * f_X$ we get

$$f_Y * f_x(w) = \int_{-\infty}^{+\infty} f_Y(z) f_X(w-z)\, \mathrm{d}z \quad \text{for all } w \in \mathbb{R}.$$

Next, make the change of variables $u = w - z$ to get $z = w - u$, $\mathrm{d}z = -\ \mathrm{d}u$ and

$$f_Y * f_x(w) = -\int_{+\infty}^{-\infty} f_Y(w-u) f_X(u)\, \mathrm{d}u \quad \text{for all } w \in \mathbb{R},$$

or

$$f_Y * f_x(w) = \int_{-\infty}^{+\infty} f_X(u) f_Y(w-u)\, \mathrm{d}u \quad \text{for all } w \in \mathbb{R},$$

which is the definition of $f_X * f_Y(w)$. Thus, we had verifies the symmetry of the convolution operation. $\square$

4. Suppose that the pdf of a random variable, W, is the convolution of two pdfs f_X and f_Y for two random variables, X and Y.

Verify that

$$M_W(t) = M_X(t) \cdot M_Y(t)$$

for t in some interval around 0 where the mgfs of X and Y are both defined; that is, the moment generating function of a convolution is the product of the moment generating functions.

Solution: Using the definition of the mgf we have that

$$\begin{aligned} M_W(t) &= \int_{-\infty}^{\infty} e^{tw} f_W(w)\ \mathrm{d}w \\ &= \int_{-\infty}^{\infty} e^{tw} f_X * f_Y(w)\ \mathrm{d}w \\ &= \int_{-\infty}^{\infty} e^{tw} \int_{-\infty}^{\infty} f_X(u) f_Y(w-u)\ \mathrm{d}u\ \mathrm{d}w \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{tw} f_X(u) f_Y(w-u)\ \mathrm{d}u\ \mathrm{d}w. \end{aligned}$$

Next, make the change of variables $u = x$ and $w - u = y$ and apply the change of variables formula to get that

$$M_W(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t(x+y)} f_X(x) f_Y(y) \left|\frac{\partial(u,v)}{\partial(x,y)}\right|\ \mathrm{d}x\ \mathrm{d}y,$$

where

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1.$$

It then follows that

$$\begin{aligned} M_W(t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{tx} f_X(x)\ e^{ty} f_Y(y)\ \mathrm{d}x\ \mathrm{d}y \\ &= \int_{-\infty}^{\infty} e^{tx} f_X(x)\ \mathrm{d}x \int_{-\infty}^{\infty} e^{ty} f_Y(y)\ \mathrm{d}y \\ &= M_X(t) M_Y(t), \end{aligned}$$

which was to be shown. □

5. Let α and β denote positive real numbers and define $f(x) = Cx^{\alpha-1}e^{-x/\beta}$ for $x > 0$ and $f(x) = 0$ for $x \leqslant 0$, where C denotes a positive real number.

   (a) Find the value of C so that f is the pdf for some distribution.

   (b) For the value of C found in part (a), let f denote the pdf of a random variable X. Compute the mgf of X.

Hint: The pdf found in part (a) is related to the Gamma function.

Solution:

(a) We choose C so that

$$\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = 1,$$

or

$$C\int_{-\infty}^{\infty} x^{\alpha-1}e^{-x/\beta}\ \mathrm{d}x = 1,$$

where, making the change of variables $t = x/\beta$, we have that $x = \beta t$, $\mathrm{d}x = \beta\ \mathrm{d}t$ and

$$\begin{aligned}\int_{-\infty}^{\infty} x^{\alpha-1}e^{-x/\beta}\ \mathrm{d}x &= \int_{-\infty}^{\infty} \beta^{\alpha-1}t^{\alpha-1}e^{-t}\beta\ \mathrm{d}t \\ &= \beta^{\alpha}\int_{-\infty}^{\infty} t^{\alpha-1}e^{-t}\ \mathrm{d}t \\ &= \beta^{\alpha}\Gamma(\alpha).\end{aligned}$$

It then follows that

$$C = \frac{1}{\beta^{\alpha}\Gamma(\alpha)}.$$

(b) Compute

$$\begin{aligned}M_X(t) &= \int_{-\infty}^{\infty} e^{tx}f(x)\ \mathrm{d}x \\ &= C\int_{0}^{\infty} e^{tx}x^{\alpha-1}e^{-x/\beta}\ \mathrm{d}x \\ &= C\int_{0}^{\infty} x^{\alpha-1}e^{-(1-\beta t)x/\beta}\ \mathrm{d}x.\end{aligned}$$

Thus, we must require that $1 - \beta t > 0$, or

$$t < \frac{1}{\beta}.$$

Next, make the change of variables $\dfrac{(1-\beta t)x}{\beta} = z$ to get that

$$x = \frac{\beta z}{1-\beta t} \quad \text{and} \quad \mathrm{d}x = \frac{\beta}{1-\beta t}\ \mathrm{d}z,$$

so that

$$\begin{aligned} M_X(t) &= C\int_0^\infty \frac{\beta^{\alpha-1}}{(1-\beta t)^{\alpha-1}}\ z^{\alpha-1}\ e^{-z}\frac{\beta}{1-\beta t}\ \mathrm{d}z \\ &= \frac{C\beta^\alpha}{(1-\beta t)^\alpha}\int_0^\infty z^{\alpha-1}\ e^{-z}\ \mathrm{d}z \\ &= \frac{C\beta^\alpha\Gamma(\alpha)}{(1-\beta t)^\alpha} \\ &= \frac{1}{(1-\beta t)^\alpha}, \end{aligned}$$

for $t < \dfrac{1}{\beta}$.

$\square$