Assignment #4

Due on Friday, September 18, 2009

Read Section 2.2 on *Transformations: Bivariate Random Variables,* pp. 84–92, in Hogg, Craig and McKean.

Background and Definitions

Convolution Formula. Recall that the convolution of the two pdfs f_X and f_Y, denoted by $f_X * f_Y$, as defined by the formula

$$f_X * f_Y(w) = \int_{-\infty}^{+\infty} f_X(u) f_Y(w-u)\ \mathrm{d}u \quad \text{for all } w \in \mathbb{R}. \tag{1}$$

If X and Y are independent, then $f_X * f_Y$ gives the pdf of the sum, $X + Y$, of the random variables X and Y.

Do the following problems

1. Suppose a system has a main component and a back–up component. The lifetime of each component may be modeled by an exponential random variable with parameter β. Let X denote the lifetime of the main component and Y the lifetime of the back–up component. Then, $X \sim$ exponential(β) and $Y \sim$ exponential(β). We may also assume that X and Y are independent random variables. The system operates as long as one of the components is working. It then follows that the total lifetime, T, of the system is the sum of X and Y. Give the distribution for T. What is the expected lifetime of the system?

2. Given real numbers a and b, with $a < b$, a random variable, X, is said to have a uniform(a, b) if the pfd of X is given by

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a < x < b; \\[2ex] 0 & \text{otherwise.} \end{cases}$$

   Suppose that X and Y are independent uniform$(0, 1)$ random variable and define $W = X + Y$. Find the pdf of W and sketch its graph.

3. Assume that X and Y are independent, continuous random variable with pdfs f_X and f_Y, respectively. Define W to be the ratio Y/X.

   Verify that the pdf of W is given by

   $$f_W(w) = \int_{-\infty}^{\infty} |u| f_X(u) f_Y(wu)\, \mathrm{d}u. \tag{2}$$

   Suggestion: First compute the cdf $F_W(w) = P\left(\frac{Y}{X} \leqslant w\right)$, and then make an appropriate change of variables.

4. Assume that X and Y are independent normal$(0,1)$ random variables and define $W = Y/X$. Use the formula (2) derived in Problem 3 to compute the pdf of W. What is the expected value of W?

5. Assume that X and Y are independent uniform$(0,1)$ random variables and define $W = Y/X$. Use the formula (2) derived in Problem 3 to compute the pdf of W. What is the expected value of W?