Solutions to Assignment #4

1. Suppose a system has a main component and a back–up component. The lifetime of each component may be modeled by an exponential random variable with parameter β. Let X denote the lifetime of the main component and Y the lifetime of the back–up component. Then, $X \sim$ exponential(β) and $Y \sim$ exponential(β). We may also assume that X and Y are independent random variables. The system operates as long as one of the components is working. It then follows that the total lifetime, T, of the system is the sum of X and Y. Give the distribution for T. What is the expected lifetime of the system?

   ***Solution**:* Since X and Y are independent, $f_{_T}(t) = f_{_X} * f_{_Y}(t)$ or

$$f_{_T}(t) = \int_{-\infty}^{\infty} f_{_X}(u) f_{_Y}(t-u)\ \mathrm{d}u,$$

   where

$$f_{_X}(x) = \begin{cases} \dfrac{1}{\beta}\ e^{-x/\beta} & \text{if } \ x > 0; \\ \\ 0 & \text{otherwise,} \end{cases}$$

   and

$$f_{_Y}(y) = \begin{cases} \dfrac{1}{\beta}\ e^{-y/\beta} & \text{if } \ y > 0; \\ \\ 0 & \text{otherwise.} \end{cases}$$

   It then follows that, for $t > 0$,

$$\begin{aligned} f_{_T}(t) &= \int_0^t \frac{1}{\beta^2}\ e^{-u/\beta}\ e^{-(t-u)/\beta}\ \mathrm{d}u \\ &= \frac{e^{-t/\beta}}{\beta^2} \int_0^t \mathrm{d}u \\ &= \frac{t\ e^{-t/\beta}}{\beta^2}. \end{aligned}$$

   We then have that

$$f_{_T}(t) = \begin{cases} \dfrac{1}{\beta^2}\ t\ e^{-t/\beta} & \text{if } \ t > 0; \\ \\ 0 & \text{if } \ t \leqslant 0. \end{cases}$$

The expected value of T is

$$E(T) = E(X) + E(Y) = 2\beta.$$

$\square$

2. Given real numbers a and b, with $a < b$, a random variable, X, is said to have a uniform(a, b) if the pfd of X is given by

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a < x < b; \\ \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that X and Y are independent uniform$(0, 1)$ random variable and define $W = X + Y$. Find the pdf of W and sketch its graph.

***Solution**:* Since X and Y are independent, we have that $f_W(w) = f_X * f_Y(w)$ or

$$f_W(w) = \int_{-\infty}^{\infty} f_X(u) f_Y(w-u)\, \mathrm{d}u,$$

where

$$f_X(x) = \begin{cases} 1 & \text{if } \ 0 < x < 1; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_Y(y) = \begin{cases} 1 & \text{if } \ 0 < y < 1; \\ 0 & \text{otherwise.} \end{cases}$$

It then follows that, for $w > 0$,

$$f_W(w) = \int_0^1 f_Y(w-u)\, \mathrm{d}u.$$

Observe that if $w \geqslant 2$, then $w - u \geqslant 1$ for all u in (0,1). It the follows that $f_Y(w-u) = 0$ for all $w \geqslant 2$ and $0 < u < 1$. Consequently,

$$f_W(w) = 0 \quad \text{for } \ w \geqslant 2.$$

We also have that

$$f_W(w) = 0 \quad \text{for} \quad w \leqslant 0.$$

It remains to see what the values of $f_W(w)$ are for $0 < w < 2$.
We consider the cases $0 < w \leqslant 1$ and $1 < w < 2$ separately. If $0 < w < 1$, write

$$\begin{aligned} f_W(w) &= \int_0^w f_Y(w-u)\, \mathrm{d}u + \int_w^1 f_Y(w-u)\, \mathrm{d}u \\ &= \int_0^w f_Y(w-u)\, \mathrm{d}u, \end{aligned}$$

since $w - u < 0$ for $w < u < 1$. It then follows that, for $0 < w < 1$,

$$f_W(w) = \int_0^w \mathrm{d}u = w,$$

since $0 < w - u < 1$ for $0 < u < w$.
Next, suppose that $1 < w < 2$ and write

$$\begin{aligned} f_W(w) &= \int_0^{w-1} f_Y(w-u)\, \mathrm{d}u + \int_{w-1}^1 f_Y(w-u)\, \mathrm{d}u \\ &= \int_{w-1}^1 f_Y(w-u)\, \mathrm{d}u, \end{aligned}$$

since $w - u > 1$ for $0 < u < w - 1$. Observing that $0 < w - u < 1$ for $w - 1 < u < 1$ and $w \geqslant 2$, we get that

$$f_W(w) = \int_{w-1}^1 \mathrm{d}u = 2 - w.$$

To summarize the calculations, we write

$$f_W(w) = \begin{cases} 0 & \text{if } w \leqslant 0; \\ w & \text{if } 0 < w \leqslant 1; \\ 2 - w & \text{if } 1 < w \leqslant 2; \\ 0 & \text{if } w > 2. \end{cases}$$

A graph of f_W is shown in Figure 1

□

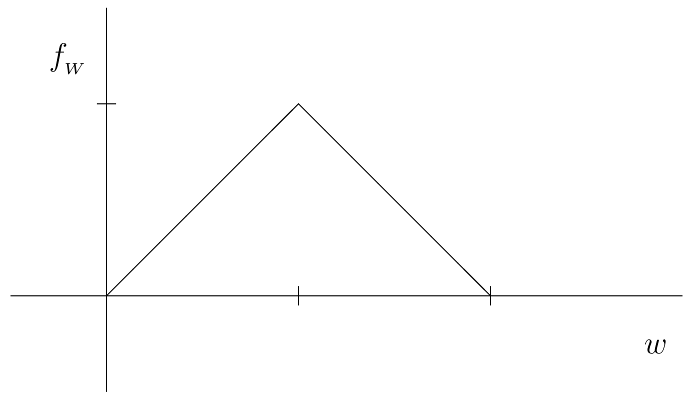

Figure 1: Graph of f_W

3. Assume that X and Y are independent, continuous random variable with pdfs f_X and f_Y, respectively. Define W to be the ratio Y/X.

   Verify that the pdf of W is given by

$$f_W(w) = \int_{-\infty}^{\infty} |u| f_X(u) f_Y(wu) \, \mathrm{d}u. \tag{1}$$

   Suggestion: First compute the cdf $F_W(w) = P\left(\frac{Y}{X} \leqslant w\right)$, and then make an appropriate change of variables.

   ***Solution**:* Compute the cdf

$$\begin{aligned} F_W(w) &= P\left(\frac{Y}{X} \leqslant w\right) \\ &= \iint_{R_w} f_{(X,Y)}(x, y) \, \mathrm{d}x \, \mathrm{d}y, \end{aligned}$$

   where the region R_w is the set defined by

$$R_w = \{(x, y) \in \mathbb{R}^2 \mid \frac{y}{x} \leqslant w, -\infty < x < \infty\},$$

   and

$$f_{(X,Y)}(x, y) = f_X(x) \cdot f_Y(y),$$

   by the assumption of independence.

Make the change of variables

$$\begin{cases} u &= x \\ v &= \dfrac{y}{x}, \end{cases}$$

so that

$$\begin{cases} x &= u \\ y &= uv, \end{cases}$$

and

$$F_W(w) = \int_{-\infty}^{w} \int_{-\infty}^{\infty} f_X(u) f_Y(vu) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, du,$$

where the Jacobian of the change of variable is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} 1 & 0 \\ v & u \end{pmatrix} = u.$$

Consequently,

$$F_W(w) = \int_{-\infty}^{w} \int_{-\infty}^{\infty} f_X(u) f_Y(vu) \, |u| \, du \, dv.$$

Differentiating with respect to w and applying the Fundamental Theorem of Calculus we obtain (1), which was to be shown. □

4. Assume that X and Y are independent normal$(0, 1)$ random variables and define $W = Y/X$. Use the formula (1) derived in Problem 3 to compute the pdf of W. What is the expected value of W?

***Solution**:* In this case,

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \, e^{-x^2/2} \quad \text{for} \quad x \in \mathbb{R},$$

and

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \, e^{-y^2/2} \quad \text{for} \quad y \in \mathbb{R}.$$

Using (1) we then have that

$$\begin{aligned} f_W(w) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |u|\ e^{-u^2/2}\ e^{-w^2u^2}/2\ \mathrm{d}u \\ &= \frac{1}{\pi}\int_{0}^{\infty} u\ e^{-(1+w^2)u^2/2}\ \mathrm{d}u, \end{aligned}$$

by the symmetry of the integrand.

Next, make the change of variable

$$z = \frac{1}{2}(1+w^2)u^2;$$

then

$$\mathrm{d}z = (1+w^2)u\ \mathrm{d}u,$$

so that

$$u\ \mathrm{d}u = \frac{1}{1+w^2}\ \mathrm{d}z$$

and

$$f_W(w) = \frac{1}{\pi}\frac{1}{1+w^2}\int_{0}^{\infty} e^{-z}\ \mathrm{d}u = \frac{1}{\pi}\frac{1}{1+w^2} \quad \text{for } w \in \mathbb{R}.$$

Observe that

$$\int_{-\infty}^{\infty} |w|\ f_W(w)\ \mathrm{d}w = \infty;$$

therefore, the expectation of W is not defined. □

5. Assume that X and Y are independent uniform$(0,1)$ random variables and define $W = Y/X$. Use the formula (1) derived in Problem 3 to compute the pdf of W. What is the expected value of W?

***Solution**:* Proceed as in the previous problem with

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_Y(y) = \begin{cases} 1 & \text{if } 0 < y < 1; \\ 0 & \text{otherwise,} \end{cases}$$

Then, applying the formula in (1),

$$f_W(w) \;=\; \int_0^1 u\, f_Y(wu)\, \mathrm{d}u$$

since $f_X(u) = 0$ for $u \leqslant 0$ or $u \geqslant 1$.
Observe that, if $w \leqslant 0$, then $f_Y(wu) = 0$ for all u with $0 < u < 1$; consequently,

$$f_W(w) \;=\; 0 \quad \text{for all} \quad w \leqslant 0.$$

We next consider the cases $0 < w \leqslant 1$ and $w > 1$ separately.
If $0 < w \leqslant 1$, then $wu < 1$ for all u in the interval $(0, 1)$; thus,

$$f_W(w) \;=\; \int_0^1 u\, \mathrm{d}u = \frac{1}{2} \quad \text{for} \quad 0 < w \leqslant 1.$$

For the case $w > 1$ observe that $\dfrac{1}{w} < 1$ and write

$$\begin{aligned} f_W(w) \;&=\; \int_0^{1/w} u\, f_Y(wu)\, \mathrm{d}u + \int_{1/w}^1 u\, f_Y(wu)\, \mathrm{d}u \\ &=\; \int_0^{1/w} u\, f_Y(wu)\, \mathrm{d}u, \end{aligned}$$

since $u > \dfrac{1}{w}$ implies that $wu > 1$. Consequently,

$$f_W(w) \;=\; \int_0^{1/w} u\, \mathrm{d}u = \frac{1}{2w^2} \quad \text{for} \quad w > 1.$$

we then have that the pdf of W is

$$f_W(w) = \begin{cases} 0 & \text{if } w \leqslant 0; \\ \dfrac{1}{2} & \text{if } 0 < w \leqslant 1; \\ \dfrac{1}{2w^2} & \text{if } w > 1. \end{cases}$$

Observe that

$$\int_{-\infty}^{\infty} w\, f_W(w)\, \mathrm{d}w = \int_0^1 \frac{w}{2}\, \mathrm{d}w + \int_1^{\infty} \frac{1}{2w}\, \mathrm{d}w = \infty,$$

and therefore the expectation of W is not defined. □