Assignment #5

Due on Friday, September 25, 2009

Read Section 5.1 on *Sampling and Statistics,* pp. 233–236, in Hogg, Craig and McKean.

Do the following problems

1. Let X denote a random variable having a normal(μ, σ^2) distribution. Define
$$Z = \frac{X - \mu}{\sigma}.$$
Compute the mgf of Z and use it to deduce the distribution of Z.

2. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a normal(μ, σ^2) distribution. Define
$$Z = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}},$$
where $\overline{X}_n$ denotes the sample mean. Compute the mgf of Z_n and use it to deduce the distribution of Z_n.

3. Let $Z \sim$ normal$(0, 1)$ and define $X = \mu + \sigma Z$. Compute the mgf of X and use it to deduce the distribution of X.

4. Let $\beta > 0$ and $X_1, X_2, \ldots, X_n$ be a random sample from an exponential(β) distribution.
Define $Y_n = \dfrac{2n\overline{X}_n}{\beta}$, where $\overline{X}_n$ is the sample mean
   (a) Determine the distribution of Y_n.
   (b) For $n = 10$, find values of c and d so that
$$\mathrm{P}\left(c < \frac{2n\overline{X}_n}{\beta} < d\right) \doteq 0.95.$$
   Use this result to give a 95% confidence interval for β based on the sample mean.

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with mean λ. Thus, $Y = \sum_{i=1}^{n} X_i$ has a Poisson distribution with mean $n\lambda$. Moreover, by the Central Limit Theorem, $\overline{X} = Y/n$ has, approximately, a normal$(\lambda, \lambda/n)$ distribution for large n.

   (a) Give the distribution of approximate distribution of

   $$\frac{Y/n - \lambda}{\sqrt{\lambda}/\sqrt{n}}$$

   for large values of n.

   (b) By the weak law of large numbers $|Y/n - \lambda|$ is very close to 0 for large values of n with a very high probability (i.e., probability very close to 1). Use this fact to obtain the approximation

   $$\sqrt{Y/n} \approx \sqrt{\lambda} + \frac{1}{2\sqrt{\lambda}}(Y/n - \lambda)$$

   for large values of n and very high probability.

   (c) Prove that, for large values of n,

   $$\mathrm{P}\left(2\sqrt{n}\left(\sqrt{Y/n} - \sqrt{\lambda}\right) \leqslant z\right) \approx \mathrm{P}(Z \leqslant z) \quad \text{for all} \quad z \in \mathbb{R}.$$

   (d) Explain how you would use the result of part (c) to obtain a confidence interval estimate for the parameter λ.