Solutions to Assignment #6

1. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a normal(μ, σ^2) distribution, where μ and σ^2 are unknown. Suppose that $n = 17$ and that the values of $X_1, X_2, \ldots, X_n$ add up to 79.90 and that the sum of the square difference from the values to the sample mean is 92.16. Give a 90% confidence interval for μ.

   ***Solution**:* The sample mean for the data is $\overline{X}_n = 4.7$ and the sample variance is $S_n^2 = 5.76$, so that $S_n = 2.4$. Thus, a 90% confidence interval for μ is given by

   $$\left(4.7 - t_{\alpha/2}\frac{2.4}{\sqrt{17}}, 4.7 + t_{\alpha/2}\frac{2.4}{\sqrt{17}}\right),$$

   where $\alpha = 0.1$ in this case. Thus, using MS Excel, R or looking up values in a table we obtain that $t_{\alpha/2} \approx 1.75$, where we have used the fact that the number of degrees of freedom is 16 in this case. Thus, a 90% confidence interval for the mean is $(4.7 - 1.02, 4.7 + 1.02)$, or $(3.68, 5.72)$. $\square$

2. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a normal(μ, σ^2) distribution, and let S_n^2 denote the sample variance. Since S_n^2 is an unbiased estimator for σ^2, $E(S_n^2) = \sigma^2$. Compute var(S_n^2); that is, compute the variance of the sampling distribution of S_n^2.

   Suggestion: Use the knowledge that you have about the distribution of $\dfrac{(n-1)}{\sigma^2}S_n^2$.

   ***Solution**:* Using the fact that the variance of a $\chi^2(n-1)$ distribution is $2(n-1)$ we obtain that

   $$\operatorname{var}\left(\frac{(n-1)}{\sigma^2}S_n^2\right) = 2(n-1),$$

   since

   $$\frac{(n-1)}{\sigma^2}S_n^2 \sim \chi^2(n-1).$$

   It then follows that

   $$\frac{(n-1)^2}{\sigma^4}\operatorname{var}(S_n^2) = 2(n-1),$$

from which we get that

$$\text{var}(S_n^2) = \frac{2\sigma^4}{n-1}.$$

$\square$

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with finite variance, σ^2. Show that

$$E(S_n) \leqslant \sigma,$$

where S_n denotes the positive square root of the sample variance.

Furthermore, prove that $E(S_n) < \sigma$ if $\text{var}(S_n) \neq 0$.

***Solution**:* Observe that

$$0 \leqslant \text{var}(S_n) = E(S_n^2) - [E(S_n)]^2 = \sigma^2 - [E(S_n)]^2.$$

It then follows that

$$[E(S_n)]^2 \leqslant \sigma^2,$$

from which the result follows. Note that if $\text{var}(S_n) \neq 0$, we get strict inequality. $\square$

4. Suppose we are sampling from a Bernoulli(p) distribution. Approximately, what should the sample size, n, be so that a 90% confidence interval for the parameter p has length at most 0.02.

***Solution**:* An approximate 90% confidence interval for p, based on the Central Limit Theorem, is based on the approximation

$$\text{P}\left(|\widehat{p}_n - p| < z_{\alpha/2}\frac{\sqrt{p(1-p)}}{\sqrt{n}}\right) \approx \text{P}(|Z| < z_{\alpha/2}) = 0.90$$

for large values of n. We then have that $z_{\alpha/2} \approx 1.65$. Observe that $p(1-p)$ is at most $1/4$. Consequently, the length of the interval is at most

$$2\ z_{\alpha/2}\frac{1}{2\sqrt{n}} = z_{\alpha/2}\frac{1}{\sqrt{n}}.$$

We want this length to be at most 0.02. Therefore

$$z_{\alpha/2}\frac{1}{\sqrt{n}} < 0.02,$$

from which we get that

$$\frac{\sqrt{n}}{z_{\alpha/2}} > 50,$$

or

$$n > (50 z_{\alpha/2})^2.$$

Hence, n should be at least $6,807$. $\square$

5. Suppose we are sampling from a Poisson(λ) distribution. A sample of 200 observations from this distribution has mean equal to 3.4. Construct and approximate 90% confidence interval for λ.

   ***Solution**:* We may use the result obtained in Problem 5 of Assignment #5:

$$\left(\left(\sqrt{\frac{Y}{n}} - \frac{z_{\alpha/2}}{2\sqrt{n}}\right)^2, \left(\sqrt{\frac{Y}{n}} + \frac{z_{\alpha/2}}{2\sqrt{n}}\right)^2\right)$$

   where Y/n is the sample mean, which in this case is 3.4 and $z_{\alpha/2}$ is 1.65. This yields an approximate 90% confidence interval for λ to be $(3.2, 3.6)$. $\square$