Solutions to Assignment #7

1. Assume that a random variable, T, has a t distribution with n degrees of freedom. Define $X = T^2$. Determine the distribution of X.

***Solution**:* First, compute the cdf of T:

$$\begin{aligned} F_X(x) &= P(X \leqslant x), \quad \text{for } x > 0, \\ &= P(T^2 \leqslant x) \\ &= P(|T| \leqslant \sqrt{x}) \\ &= P(-\sqrt{x} \leqslant T \leqslant \sqrt{x}) \\ &= P(-\sqrt{x} < T \leqslant \sqrt{x}), \end{aligned}$$

where we have used the fact that T is a continuous random variable. Thus,

$$F_X(x) = F_T(\sqrt{x}) - F_T(-\sqrt{x}), \quad \text{for } x > 0.$$

Differentiating with respect to x yields

$$f_X(x) = f_T(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} + f_T(-\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}, \quad \text{for } x > 0,$$

where we have used the chain rule. Consequently, by the symmetry of the pdf for the T distribution,

$$f_X(x) = \frac{1}{\sqrt{x}} f_T(\sqrt{x}), \quad \text{for } x > 0,$$

where

$$f_T(t) = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)} \frac{1}{\sqrt{n\pi}} \frac{1}{(1 + (t^2/n))^{(n+1)/2}}, \quad \text{for } -\infty < t < \infty.$$

It then follows that

$$f_X(x) = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)} \frac{1}{\sqrt{n\pi}} \frac{x^{-1/2}}{(1 + (x/n))^{(n+1)/2}}, \quad \text{for } x > 0,$$

or

$$f_X(x) = \frac{\Gamma((n+1)/2)}{\Gamma(1/2)\Gamma(n/2)} \left(\frac{1}{n}\right)^{1/2} \frac{x^{-1/2}}{(1 + (x/n))^{(n+1)/2}}, \quad \text{for } x > 0,$$

which is the pdf of an $F(1, n)$ random variable. Consequently, $X = T^2$ has an $F(1, n)$ distribution. □

2. Recall that in Problem 3 of Assignment #4 you verified that if X and Y are independent random variables with pdfs f_X and f_Y, respectively, and $W = Y/X$, then the pdf of W is given by

$$f_W(w) = \int_{-\infty}^{\infty} |u| f_X(u) f_Y(wu)\, du. \tag{1}$$

Suppose that X and Y are independent exponential(1) random variables and define $W = Y/X$. Compute the pdf of W and determine the type of distribution that W has.

***Solution**:* The pdfs of X and Y are

$$f_X(x) = \begin{cases} e^{-x} & \text{if } x > 0; \\ 0 & \text{if } x \leqslant 0, \end{cases}$$

and

$$f_Y(y) = \begin{cases} e^{-y} & \text{if } y > 0; \\ 0 & \text{if } y \leqslant 0, \end{cases}$$

respectively. We then have that

$$f_W(w) = \int_0^{\infty} u\, e^{-u} f_Y(wu)\, du.$$

Thus, for $w \leqslant 0$, $f_W(w) = 0$, and, for $w > 0$,

$$\begin{aligned} f_W(w) &= \int_0^{\infty} u\, e^{-u} e^{-wu}\, du \\ &= \int_0^{\infty} u\, e^{-(1+w)u}\, du. \end{aligned}$$

Integration by parts then yields

$$f_W(w) = \frac{1}{(1+w)^2} \quad \text{for} \quad w > 0,$$

which is the pdf of an $F(2,2)$ random variable. Hence, W has an $F(2,2)$ distribution. □

3. Let $X \sim \chi^2(n-1)$ and $Y \sim \chi^2(m-1)$ be independent random variables and define $W = \dfrac{Y/(m-1)}{X/(n-1)}$. Use the formula in (1) to compute the pdf of W. Determine the type of distribution that W has.

***Solution**:* We first determine the pdfs of $Y/(m-1)$ and $X/(n-1)$ so we can apply formula (1). Write $U = X/(n-1)$. Then, the cdf of U is

$$\begin{aligned} F_U(u) &= \mathrm{P}\,(U \leqslant u) \\ &= \mathrm{P}\left(\frac{X}{n-1} \leqslant u\right) \\ &= \mathrm{P}\,(\leqslant (n-1)u) \\ &= F_X((n-1)u). \end{aligned}$$

Differentiating with respect to u we then obtain that

$$f_U(u) = (n-1)f_X((n-1)u).$$

Thus the pdf on $X/(n-1)$ is

$$f_U(u) = (n-1)\frac{1}{\Gamma((n-1)/2)2^{(n-1)/2}}[(n-1)u]^{((n-1)/2)-1}e^{-(n-1)u/2},$$

for $u > 0$, which we can re–write us

$$f_U(u) = \frac{(n-1)^{(n-1)/2}}{\Gamma((n-1)/2)2^{(n-1)/2}}u^{((n-1)/2)-1}e^{-(n-1)u/2},$$

for $u > 0$, and 0 for $u \leqslant 0$. Similarly, the pdf for $V = Y/(m-1)$ is

$$f_V(v) = \frac{(m-1)^{(m-1)/2}}{\Gamma((m-1)/2)2^{(m-1)/2}}v^{((m-1)/2)-1}e^{-(m-1)v/2},$$

for $v > 0$ and 0 for $v \leqslant 0$. Using the formula in (1) we then have that the pdf for $W = V/U$ is

$$\begin{aligned} f_W(w) &= \int_0^\infty u\,\frac{(n-1)^{(n-1)/2}}{\Gamma((n-1)/2)2^{(n-1)/2}}u^{((n-1)/2)-1}e^{-(n-1)u/2}f_V(wu)\,\mathrm{d}u \\ &= \int_0^\infty \frac{(n-1)^{(n-1)/2}}{\Gamma((n-1)/2)2^{(n-1)/2}}u^{(n-1)/2}e^{-(n-1)u/2}f_V(wu)\,\mathrm{d}u. \end{aligned}$$

Then, for $w \leqslant 0$, $f_{_W}(w) = 0$ and, for $w > 0$

$$f_{_W}(w) \;=\; C_{m,n}\int_0^\infty u^{(n-1)/2}e^{-(n-1)u/2}(wu)^{((m-1)/2)-1}e^{-(m-1)wu/2}\,\mathrm{d}u,$$

where the constant $C_{m,n}$ is given by

$$C_{m,n} = \frac{(n-1)^{(n-1)/2}(m-1)^{(m-1)/2}}{\Gamma((n-1)/2)\Gamma((m-1)/2)2^{(n-1)/2}2^{(m-1)/2}}.$$

Thus,

$$f_{_W}(w) \;=\; C_{m,n}w^{(\nu_1/2)-1}\int_0^\infty u^{(\nu_1+\nu_2)/2-1}e^{-(\nu_2+\nu_1 w)u/2}\,\mathrm{d}u,$$

where we have written ν_1 for $m-1$ and ν_2 for $n-1$. Next, make the change of variables $z = (\nu_2+\nu_1 w)u/2$, so that $u = \dfrac{2}{\nu_2+\nu_1 w}z$ and

$$\begin{aligned} f_{_W}(w) &= C_{m,n}\frac{2^{(\nu_1+\nu_2)/2}}{(\nu_2+\nu_1 w)^{(\nu_1+\nu_2)/2}}w^{(\nu_1-2)/2}\int_0^\infty z^{(\nu_1+\nu_2)/2-1}e^{-z}\,\mathrm{d}z \\ &= C_{m,n}\frac{2^{(\nu_1+\nu_2)/2}\Gamma((\nu_1+\nu_2)/2)}{(\nu_2+\nu_1 w)^{(\nu_1+\nu_2)/2}}\;w^{(\nu_1-2)/2}. \end{aligned}$$

Observe that $C_{m,n}$ can be written in terms of ν_1 and ν_2 as follows:

$$C_{m,n} = \frac{\nu_1^{\nu_1/2}\nu_2^{\nu_2/2}}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)2^{\nu_1/2}2^{\nu_2/2}}.$$

It then follows that

$$\begin{aligned} f_{_W}(w) &= \frac{\Gamma((\nu_1+\nu_2)/2)}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)}\;\frac{\nu_1^{\nu_1/2}\nu_2^{\nu_2/2}}{\nu_2^{(\nu_1+\nu_2)/2}}\frac{w^{(\nu_1-2)/2}}{(1+\nu_1 w/\nu_2)^{(\nu_1+\nu_2)/2}} \\ &= \frac{\Gamma((\nu_1+\nu_2)/2)}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)}\;\frac{\nu_1^{\nu_1/2}}{\nu_2^{\nu_1/2}}\frac{w^{(\nu_1-2)/2}}{(1+\nu_1 w/\nu_2)^{(\nu_1+\nu_2)/2}}, \end{aligned}$$

which is the pdf of an $F(\nu_1,\nu_2)$ random variable. It then follows that $W = \dfrac{Y/(m-1)}{X/(n-1)}$ has an $F(m-1,n-1)$ distribution. □

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal$(\mu_{_X}, \sigma^2)$ distribution and $Y_1, Y_2, \ldots, Y_m$ be a random sample from a normal$(\mu_{_Y}, \sigma^2)$. Let $S^2_{_X}$ denote the sample variance of the random sample $X_1, X_2, \ldots, X_n$ and $S^2_{_Y}$ that of the random sample $Y_1, Y_2, \ldots, Y_m$. Determine the distribution of $S^2_{_Y}/S^2_{_X}$ and use that information to show how to find $\mathrm{P}\left(\dfrac{S^2_{_Y}}{S^2_{_X}} > c\right)$ for any $c > 0$.

***Solution**:* Write $\dfrac{S^2_{_Y}}{S^2_{_X}} = \dfrac{\dfrac{1}{\sigma^2}S^2_{_Y}}{\dfrac{1}{\sigma^2}S^2_{_X}}$ and note that

$$\frac{m-1}{\sigma^2}S^2_{_Y} \sim \chi^2(m-1) \quad \text{and} \quad \frac{n-1}{\sigma^2}S^2_{_X} \sim \chi^2(n-1).$$

Putting $V = \dfrac{m-1}{\sigma^2}S^2_{_Y}$ and $U = \dfrac{n-1}{\sigma^2}S^2_{_X}$, we see that $\dfrac{S^2_{_Y}}{S^2_{_X}} = \dfrac{V/(m-1)}{U/(n-1)}$, where

$$V \sim \chi^2(m-1) \quad \text{and} \quad U \sim \chi^2(n-1).$$

Since V and U are independent, as they come from two independent random samples, it follows from Problem 3 that $S^2_{_Y}/S^2_{_X}$ has an $F(m-1, n-1)$ distribution.

Knowing m and n, we can then use an F distribution table, or some statistical software, to determine $\mathrm{P}\left(\dfrac{S^2_{_Y}}{S^2_{_X}} > c\right)$ for any $c > 0$. □

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal$(\mu_{_X}, \sigma^2_{_X})$ distribution and $Y_1, Y_2, \ldots, Y_m$ be a random sample from a normal$(\mu_{_Y}, \sigma^2_{_Y})$. Let $S^2_{_X}$ denote the sample variance of the random sample $X_1, X_2, \ldots, X_n$ and $S^2_{_Y}$ that of the random sample $Y_1, Y_2, \ldots, Y_m$. Determine the distribution of $\dfrac{S^2_{_Y}/\sigma^2_{_Y}}{S^2_{_X}/\sigma^2_{_X}}$ and use that information to explain how to find a 95% confidence interval for $\sigma^2_{_Y}/\sigma^2_{_X}$.

***Solution**:* Put $V = \dfrac{m-1}{\sigma^2_{_Y}}S^2_{_Y}$ and $U = \dfrac{n-1}{\sigma^2_{_X}}S^2_{_X}$. Then,

$$\frac{S^2_{_Y}/\sigma^2_{_Y}}{S^2_{_X}/\sigma^2_{_X}} = \frac{V/(m-1)}{U/(n-1)},$$

where $V \sim \chi^2(m-1)$ and $U \sim \chi^2(n-1)$. Since V and U are independent, as they come from two independent random samples, it follows from Problem 3 that $\dfrac{S_Y^2/\sigma_Y^2}{S_X^2/\sigma_X^2}$ has an $F(m-1, n-1)$ distribution. We can then find c and d such that $c < d$ and

$$\mathrm{P}\left(\frac{S_Y^2/\sigma_Y^2}{S_X^2/\sigma_X^2} < c\right) = \frac{\alpha}{2} \quad \text{and} \quad \mathrm{P}\left(\frac{S_Y^2/\sigma_Y^2}{S_X^2/\sigma_X^2} < d\right) = 1 - \frac{\alpha}{2}.$$

Then

$$\mathrm{P}\left(c < \frac{S_Y^2/\sigma_Y^2}{S_X^2/\sigma_X^2} < d\right) = 1 - \alpha,$$

or

$$\mathrm{P}\left(c < \frac{S_Y^2/S_X^2}{\sigma_Y^2/\sigma_X^2} < d\right) = 1 - \alpha,$$

or

$$\mathrm{P}\left(\frac{1}{d} < \frac{\sigma_Y^2/\sigma_X^2}{S_Y^2/S_X^2} < \frac{1}{c}\right) = 1 - \alpha,$$

or

$$\mathrm{P}\left(\frac{S_Y^2/S_X^2}{d} < \sigma_Y^2/\sigma_X^2 < \frac{S_Y^2/S_X^2}{c}\right) = 1 - \alpha.$$

Hence, the interval

$$\left(\frac{S_Y^2/S_X^2}{d}, \ \frac{S_Y^2/S_X^2}{c}\right)$$

is a $100(1-\alpha)\%$ confidence interval for σ_Y^2/σ_X^2. The 95% confidence interval is obtained with $\alpha = 0.05$. □