Assignment #8

Due on Friday, October 16, 2009

Read Section 5.7 on *Chi–Square Tests,* pp. 278–284, in Hogg, Craig and McKean.

Background and Definitions

- **Covariance.** Given random variables X and Y, the covariance of X and Y is defined to be
$$\operatorname{cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)],$$
where $\mu_X = E(X)$ and $\mu_Y = E(Y)$.

- **Conditional Distribution.** Given two discrete random variables, X and Y, with joint probability mass function (pmf) $p_{(X,Y)}(k,\ell)$, the conditional pmf of X given Y, denoted $p_{X|Y}(\ell \mid k)$, is defined to be
$$p_{X|Y}(\ell \mid k) = \frac{p_{(X,Y)}(\ell,k)}{p_Y(k)},$$
where the marginal pdf of Y at k, $p_Y(k)$ is assumed to be positive. The distribution determined by $p_{X|Y}$ is called the conditional distribution of X, given Y.

- **Multinomial Distribution.** Let n and k denote positive integers. Let $p_1, p_2, \ldots, p_k$ denote numbers satisfying $0 \leqslant p_i \leqslant 1$ for all $i = 1, 2, \ldots, k$ and
$$\sum_{i=1}^{k} p_i = 1.$$
Suppose that $X_1, X_2, \ldots, X_k$ with discrete random variables with joint pmf given by
$$p_{(X_1,X_2,\ldots,X_k)}(n_1,n_2,\ldots,n_k) = \begin{cases} \dfrac{n!}{n_1!n_2!\cdots n_k!} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} & \text{if } \sum_{i=1}^{k} n_k = n; \\ \\ 0 & \text{otherwise,} \end{cases}$$
we then say that the random vector $(X_1, X_2, \ldots, X_k)$ has a multinomial distribution with parameters $n, p_1, p_2, \ldots, p_k$.

Do the following problems

1. Let the random vector (X_1, X_2) have a multinomial distribution with parameters n, p_1, p_2.

   (a) Give the marginal distributions for X_1 and X_2 and compute $E(X_i)$ for $i = 1, 2$.

   (b) Show that X_1 and X_2 are not independent and compute the covariance, $\text{cov}(X_1, X_2)$, of X_1 and X_2.

2. Given two random variables, X and Y, the joint moment generating function of X and Y, denoted by $M_{(X,Y)}(t_1, t_2)$, is defined to be be

$$M_{(X,Y)}(t_1, t_2) = E(e^{t_1 X + t_2 Y})$$

   for (t_1, t_2) in some neighborhood of the origin in $\mathbb{R}^2$.

   Let the random vector (X_1, X_2) have a multinomial distribution with parameters n, p_1, p_2.

   (a) Compute the joint mgf of (X_1, X_2).

   (b) Verify that $\text{cov}(X_1, X_2) = \dfrac{\partial^2 M}{\partial t_1 \partial t_2}(0, 0) - \dfrac{\partial M}{\partial t_1}(0, 0)\dfrac{\partial M}{\partial t_2}(0, 0)$, where $M = M_{(X_1, X_2)}$.

3. Let X_1 and X_2 be independent Poisson(λ) random variables. For a fixed value of n ($n = 0, 1, 2, 3, \ldots$), determine the conditional distribution of X_1 given that $X_1 + X_2 = n$.

4. Let $X_1, X_2, \ldots, X_k$ be independent random variables satisfying $X_i \sim \text{Poisson}(\lambda_i)$ for positive parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$. For a fixed value of n ($n = 0, 1, 2, 3, \ldots$), determine the conditional distribution of the random vector $(X_1, X_2, \ldots, X_k)$ given that $X_1 + X_2 + \cdots + X_k = n$.

5. Let the random vector (X_1, X_2) have a multinomial distribution with parameters n, p_1, p_2. Define the random variable $Q = \dfrac{(X_1 - np_1)^2}{np_1} + \dfrac{(X_2 - np_2)^2}{np_2}$. Show that for large values of n, Q has, approximately, a $\chi^2(1)$ distribution.

   Suggestion Use the result of part (a) in Problem 1 and apply the Central Limit Theorem.