Review Problems for Exam 2

1. In the book “Experimentation and Measurement,” by W. J. Youden and published by the National Science Teachers Association in 1962, the author reported an experiment, performed by a high school student and a younger brother, which consisted of tossing five coins and recording the frequencies for the number of heads in the five coins. The data collected are shown in Table 1.

<table>
<thead>
<tr>
<th>Number of Heads</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>100</td>
<td>524</td>
<td>1080</td>
<td>1126</td>
<td>655</td>
<td>105</td>
</tr>
</tbody>
</table>

Table 1: Frequency Distribution for a Five–Coin Tossing Experiment

(a) Are the data in Table 1 consistent with the hypothesis that all the coins were fair? Justify your answer.

(b) Assume now that the coins have the same probability, $p$, of turning up heads. Estimate $p$ and perform a goodness of fit test of the model you used to do your estimation. What do you conclude?

2. In 1,000 tosses of a coin, 560 yield heads and 440 turn up tails. Is it reasonable to assume that the coin is fair? Justify your answer.

3. In a random sample, $X_1, X_2, \ldots, X_n$, of Bernoulli($p$) random variables, it is desired to test the hypotheses $H_0: p = 0.49$ versus $H_1: p = 0.51$. Use the Central Limit Theorem to determine, approximately, the sample size, $n$, needed to have the probabilities of Type I error and Type II error to be both about 0.01. Explain your reasoning.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal($\theta, 1$) distribution. Suppose you want to test $H_0: \theta = \theta_o$ versus $H_1: \theta \neq \theta_o$, with the rejection region defined by $\sqrt{n}|\bar{X}_n - \theta_o| > c$, for some critical value $c$.

(a) Find an expression in terms of standard normal probabilities for the power function of this test.

(b) An experimenter desires a Type I error probability of 0.04 and a maximum Type II error probability of 0.25 at $\theta = \theta_o + 1$. Find the values of $n$ and $c$ for which these conditions can be achieved.
5. Let \( X_1, X_2, \ldots, X_n \) be a random sample from a normal(\( \theta, \sigma^2 \)) distribution. Suppose you want to test

\[
H_0 : \theta \leq \theta_0
\]

versus

\[
H_1 : \theta > \theta_1
\]

with the rejection region defined by

\[
T_n(\theta) > \frac{\sqrt{n}}{S_n}(\theta_o - \theta) + c,
\]

for some critical value \( c \). Here, \( T_n(\theta) \) is the statistic

\[
T_n(\theta) = \frac{\sqrt{n}(\bar{X}_n - \theta)}{S_n},
\]

where \( \bar{X}_n \) and \( S_n^2 \) are the sample mean and variance, respectively.

(a) If the significance level for the test is to be set at \( \alpha \), what should \( c \) be?

(b) Express the rejection region in terms of the value \( c \) found in part (a), and the statistics \( \bar{X}_n \) and \( S_n^2 \).

(c) Compute the power function, \( \gamma(\theta) \), for the test.

6. A sample of 16 “10–ounce” cereal boxes has a mean weight of 10.4 oz and a standard deviation of 0.85 oz. Perform an appropriate test to determine whether, on average, the “10–ounce” cereal boxes weigh something other than 10 ounces at the \( \alpha = 0.05 \) significance level. Explain your reasoning.

7. Find the \( p \)-value of observed data consisting of 7 successes in 10 Bernoulli(\( \theta \)) trials in a test of

\[
H_0 : \theta = \frac{1}{2} \quad \text{versus} \quad H_1 : \theta > \frac{1}{2}.
\]

8. Three independent observations from a Poisson(\( \lambda \)) distribution yield the values \( x_1 = 3, x_2 = 5 \) and \( x_3 = 1 \). Explain how you would use these data to test the hypothesis \( H_0 : \lambda = 1 \) versus the alternative \( H_1 : \lambda > 1 \). Come up with an appropriate statistic and rejection criterion and determine the \( p \)-value given by the data. What do you conclude?