Review Problems for Exam 3

1. Let X have a Gamma distribution with parameters $\alpha = 4$ and $\beta = \theta > 0$.

   (a) Find the Fisher information $I(\theta)$.

   (b) Let $X_1, X_2, \ldots, X_n$ be a random sample from a Gamma$(4, \theta)$ distribution. Find the MLE for θ and show that it is an efficient estimator.

2. Let X have a normal distribution with mean $\mu = 0$ and variance $\sigma^2 = \theta > 0$.

   (a) Find the Fisher information $I(\theta)$.

   (b) Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal$(0, \theta)$ distribution. Find the MLE for θ and show that it is an efficient estimator.

3. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a uniform distribution over the interval $[0, \theta]$ for some parameter $\theta > 0$.

   Show that $W = 2\overline{X}_n$ is an unbiased estimator of θ and determine its efficiency.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal$(0, \theta)$ distribution. We want to use the statistic
$$Y = \sum_{i=1}^{n} |X_i|$$
   to estimate the standard deviation $\sqrt{\theta}$.

   (a) Let $W = cY$ for some constant c. Determine a value of c so that W is an unbiased estimator of $\sqrt{\theta}$.

   (b) Compute the efficiency of the estimator W found in part (a).

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal(μ_o, θ) distribution, where μ_o is known and $\theta > 0$. Show that the LRT for
$$\mathrm{H}_o\colon \quad \theta = \theta_o \quad \text{versus} \quad \mathrm{H}_1\colon \quad \theta \neq \theta_o$$
   may be based upon the statistic
$$W = \frac{1}{\theta_o} \sum_{i=1}^{n} (X_i - \mu_o)^2.$$
   Determine the null distribution of W and give, explicitly, the rejection rule for a level α test.

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Gamma$(4, \theta)$ distribution with $\theta > 0$.

   (a) Show that the LRT for

   $$\mathrm{H}_o\colon \quad \theta = \theta_o \quad \text{versus} \quad \mathrm{H}_1\colon \quad \theta \neq \theta_o$$

   may be based upon the statistic

   $$W = \sum_{i=1}^{n} X_i.$$

   Determine the null distribution of $2W/\theta_o$.

   (b) For $\theta_o = 4$ and $n = 5$, find c_1 and c_2 so that the test rejects H_o when $W \leqslant c_1$ or $W \geqslant c_2$ has a significance level $\alpha = 0.05$.

7. Suppose that $X_1, X_2, \ldots, X_n$ form a random sample from a normal$(0, \sigma^2)$ distribution. We wish to test

   $$\mathrm{H}_o\colon \quad \sigma^2 \leqslant 2 \quad \text{versus} \quad \mathrm{H}_1\colon \quad \sigma^2 > 2.$$

   (a) Show that there exists a uniformly most powerful (UMP) test at every significance level α.

   (b) Show that the UMP test found in part (a) rejects H_o when

   $$\sum_{i=1}^{n} X_i^2 \geqslant c,$$

   for some $c > 0$, and determine the value of c so that the significance level of the test is $\alpha = 0.05$.

8. In a given city, it is assumed that the number of automobile accidents in a given year follows a Poisson distribution. Suppose that it is known that, in past years, the average number of accidents per years was 15. Suppose that this year the number of accidents has been 10. Is it justified to claim that the rate of accidents has dropped?

   To answer this question, set up an appropriate hypothesis test. State your assumptions clearly and justify your conclusions.