Lecture Examples

1. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a normal(μ, σ^2) distribution. Show that the sample mean has a normal$(\mu, \sigma^2/n)$ distribution.

   Suggestion: Use moment generating functions.

   Solution: Compute the mgf of the sample mean $\overline{X}_n = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$; namely,

$$\begin{aligned} M_{\overline{X}_n}(t) &= E(e^{t\overline{X}_n}) \\ &= E\left(e^{(X_1+X_2+\cdots+X_n)\frac{t}{n}}\right) \\ &= M_{X_1+X_2+\cdots+X_n}\left(\frac{t}{n}\right) \\ &= M_{X_1}\left(\frac{t}{n}\right) \cdot M_{X_2}\left(\frac{t}{n}\right) \cdots M_{X_n}\left(\frac{t}{n}\right) \end{aligned}$$

   where we have used the independence of the random variables $X_1, X_2, \ldots, X_n$. Next, use the assumption that they identically distributed with a normal(μ, σ^2) distribution, we obtain that

$$\begin{aligned} M_{\overline{X}_n}(t) &= \left[M_{X_1}\left(\frac{t}{n}\right)\right]^n \\ &= \left[e^{\mu\frac{t}{n}+\sigma^2\frac{(t/n)^2}{2}}\right]^n \\ &= e^{\mu t+\frac{\sigma^2}{n}t^2/2}, \end{aligned}$$

   which is the mgf of normal$(\mu, \sigma^2/n)$ random variable. It then follows that $\overline{X}_n$ has a normal$(\mu, \sigma^2/n)$ distribution. □

2. *The χ^2 distribution.*

   (a) *One degree of freedom.* Let $Z \sim$ normal$(0,1)$ and define $X = Z^2$. Find the pdf and mgf for X. Compute also the mean and variance of X. The random variable X is said to have a χ^2 distribution one degree of freedom, and we write $X \sim \chi^2(1)$.

Solution: The pdf of Z is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}, \quad \text{for } -\infty < z < \infty.$$

We compute the pdf for X by first determining its cumulative density function (cdf):

$$\begin{aligned} P(X \le x) &= P(Z^2 \le x) \quad \text{for } y \geqslant 0 \\ &= P(-\sqrt{x} \le Z \le \sqrt{x}) \\ &= P(-\sqrt{x} < Z \le \sqrt{x}), \quad \text{since Z is continuous.} \end{aligned}$$

Thus,

$$\begin{aligned} P(X \le x) &= P(Z \le \sqrt{x}) - P(Z \le -\sqrt{x}) \\ &= F_Z(\sqrt{x}) - F_Z(-\sqrt{x}) \quad \text{for } x > 0, \end{aligned}$$

since X is continuous.
We then have that the cdf of X is

$$F_X(x) = F_Z(\sqrt{x}) - F_Z(-\sqrt{x}) \quad \text{for } x > 0,$$

from which we get, after differentiation with respect to x,

$$\begin{aligned} f_X(x) &= F_Z'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} + F_Z'(-\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \\ &= f_Z(\sqrt{x})\frac{1}{2\sqrt{x}} + f_Z(-\sqrt{x})\frac{1}{2\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}\left\{\frac{1}{\sqrt{2\pi}}\, e^{-x/2} + \frac{1}{\sqrt{2\pi}}\, e^{-x/2}\right\} \\ &= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{x}}\, e^{-x/2} \end{aligned}$$

for $x > 0$.
Thus, the pdf of X is

$$f_X(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}}\dfrac{1}{\sqrt{x}}\, e^{-x/2} & x > 0, \\ \\ 0 & \text{elsewhere.} \end{cases}$$

□

Definition. *(χ^2 distribution with n degrees of freedom)* Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with a $\chi^2(1)$ distribution. Then then random variable $X_1 + X_2 + \cdots + X_n$ is said to have a χ^2 distribution with n degrees of freedom. We write

$$X_1 + X_2 + \cdots + X_n \sim \chi^2(n).$$

(b) *Two degrees of freedom.* Let X and Y be two independent random variable with a $\chi^2(1)$ distribution. We would like to know the distribution of the sum $X + Y$.

Solution: Denote the sum $X + Y$ by W. We would like to compute the pdf f_W, given that the pdfs of X and Y are

$$f_X(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \dfrac{1}{\sqrt{x}}\, e^{-x/2} & x > 0, \\ \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \dfrac{1}{\sqrt{y}}\, e^{-y/2} & y > 0 \\ \\ 0 & \text{otherwise,} \end{cases}$$

respectively.
We first compute the cdf

$$F_W(w) = P(W \leqslant w) \quad \text{for } w > 0,$$

where

$$\begin{aligned} P(W \leqslant w) &= P(X + Y \leqslant w) \\ &= \iint_{\{x+y \leqslant w\}} f_{(X,Y)}(x, y)\, \mathrm{d}x\, \mathrm{d}y. \end{aligned}$$

Since X and Y are independent, the joint pdf of X and Y is given by

$$f_{(X,Y)}(x, y) = f_X(x) \cdot f_Y(y)$$

or

$$f_{(X,Y)}(x,y) = \begin{cases} \dfrac{1}{2\pi}\ \dfrac{1}{\sqrt{x}\sqrt{y}}\ e^{-(x+y)/2} & \text{if } \ x > 0, y > 0; \\ 0 & \text{otherwise.} \end{cases}$$

We then have that, for $w > 0$,

$$F_W(w) \ = \ \frac{1}{2\pi} \int_0^w \int_0^{w-x} \frac{1}{\sqrt{x}\sqrt{y}}\ e^{-(x+y)/2}\ \mathrm{d}y\ \mathrm{d}x,$$

see Figure 1.

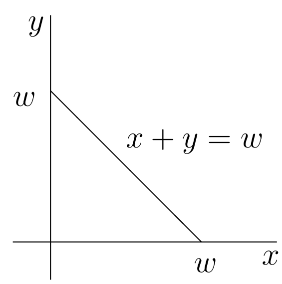

Figure 1: $\{x + y \leqslant w\}$

Next, make the change of variables: $u = x$, $v = x + y$ to get that

$$F_W(w) \ = \ \frac{1}{2\pi} \int_0^w \int_0^w \frac{1}{\sqrt{u}\sqrt{v-u}}\ e^{-v/2} \left| \frac{\partial(x,y)}{\partial(u,v)} \right|\ \mathrm{d}u\ \mathrm{d}v,$$

where the Jacobian of the change of variables is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = 1.$$

Consequently,

$$F_W(w) \ = \ \frac{1}{2\pi} \int_0^w \ e^{-v/2} \int_0^w \frac{1}{\sqrt{u}\sqrt{v-u}}\ \mathrm{d}u\ \mathrm{d}v.$$

Next, differentiate with respect to w to obtain the pdf

$$f_W(w) \ = \ \frac{1}{2\pi}\ e^{-w/2} \int_0^w \frac{1}{\sqrt{u}\sqrt{w-u}}\ \mathrm{d}u,$$

where we have applied the Fundamental Theorem of Calculus. Thus, making the change of variables $t = \dfrac{u}{w}$, so that $\mathrm{d}u = w\ \mathrm{d}t$,

$$
\begin{aligned}
f_{_W}(w) &= \frac{e^{-w/2}}{2\pi}\int_0^1 \frac{w}{\sqrt{wt}\sqrt{w-wt}}\ \mathrm{d}t \\
&= \frac{e^{-w/2}}{2\pi}\int_0^1 \frac{1}{\sqrt{t}\sqrt{1-t}}\ \mathrm{d}t.
\end{aligned}
$$

Making a second change of variables, $s = \sqrt{t}$, we get that $t = s^2$ and $\mathrm{d}t = 2s\ \mathrm{d}s$, so that

$$
\begin{aligned}
f_{_W}(w) &= \frac{e^{-w/2}}{\pi}\int_0^1 \frac{1}{\sqrt{1-s^2}}\ \mathrm{d}s \\
&= \frac{e^{-w/2}}{\pi}\left[\arcsin(s)\right]_0^1 \\
&= \frac{1}{2}\,e^{-w/2} \quad \text{for } w > 0,
\end{aligned}
$$

and zero otherwise. It then follows that $W = X + Y$ has the pdf of an exponential(2) random variable. □

(c) *Three degrees of freedom.* Let $X \sim$ exponential(2) and $Y \sim \chi^2(1)$ be independent random variables and define $W = X + Y$. Give the distribution of W.

Solution: Since X and Y are independent, by Problem 1 in Assignment #3, $f_{_W}$ is the convolution of $f_{_X}$ and $f_{_Y}$:

$$
\begin{aligned}
f_{_W}(w) &= f_{_X} * f_{_Y}(w) \\
&= \int_{-\infty}^{\infty} f_{_X}(u) f_{_Y}(w-u)\ \mathrm{d}u,
\end{aligned}
$$

where

$$
f_{_X}(x) = \begin{cases} \dfrac{1}{2}e^{-x/2} & \text{if } x > 0; \\ 0 & \text{otherwise;} \end{cases}
$$

and

$$
f_{_Y}(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi}}\dfrac{1}{\sqrt{y}}\ e^{-y/2} & \text{if } y > 0; \\ 0 & \text{otherwise.} \end{cases}
$$

It then follows that, for $w > 0$,

$$\begin{aligned} f_{_W}(w) &= \int_0^\infty \frac{1}{2} e^{-u/2} f_{_Y}(w-u)\,\mathrm{d}u \\ &= \int_0^w \frac{1}{2} e^{-u/2} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{w-u}}\ e^{-(w-u)/2}\,\mathrm{d}u \\ &= \frac{e^{-w/2}}{2\sqrt{2\pi}} \int_0^w \frac{1}{\sqrt{w-u}}\,\mathrm{d}u. \end{aligned}$$

Making the change of variables $t = u/w$, we get that $u = wt$ and $\mathrm{d}u = w\ \mathrm{d}t$, so that

$$\begin{aligned} f_{_W}(w) &= \frac{e^{-w/2}}{2\sqrt{2\pi}} \int_0^1 \frac{1}{\sqrt{w-wt}}\ w\ \mathrm{d}t \\ &= \frac{\sqrt{w}\ e^{-w/2}}{2\sqrt{2\pi}} \int_0^1 \frac{1}{\sqrt{1-t}}\ \mathrm{d}t \\ &= \frac{\sqrt{w}\ e^{-w/2}}{\sqrt{2\pi}} \left[-\sqrt{1-t}\right]_0^1 \\ &= \frac{1}{\sqrt{2\pi}}\ \sqrt{w}\ e^{-w/2}, \end{aligned}$$

for $w > 0$. It then follows that

$$f_{_W}(w) = \begin{cases} \dfrac{1}{\sqrt{2\pi}}\ \sqrt{w}\ e^{-w/2} & \text{if } w > 0; \\ 0 & \text{otherwise.} \end{cases}$$

This is the pdf for a $\chi^2(3)$ random variable. □

(d) *Four degrees of freedom.* Let $X, Y \sim$ exponential(2) be independent random variables and define $W = X + Y$. Give the distribution of W.

Solution: Since X and Y are independent, $f_{_W}$ is the convolution of $f_{_X}$ and $f_{_Y}$:

$$\begin{aligned} f_{_W}(w) &= f_{_X} * f_{_Y}(w) \\ &= \int_{-\infty}^\infty f_{_X}(u) f_{_Y}(w-u)\,\mathrm{d}u, \end{aligned}$$

where

$$f_X(x) = \begin{cases} \dfrac{1}{2} e^{-x/2} & \text{if } \; x > 0; \\ \\ 0 & \text{otherwise;} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \dfrac{1}{2}\; e^{-y/2} & \text{if } \; y > 0; \\ \\ 0 & \text{otherwise.} \end{cases}$$

It then follows that, for $w > 0$,

$$\begin{aligned} f_W(w) &= \int_0^\infty \frac{1}{2} e^{-u/2} f_Y(w-u)\; \mathrm{d}u \\ &= \int_0^w \frac{1}{2} e^{-u/2} \frac{1}{2}\; e^{-(w-u)/2}\; \mathrm{d}u \\ &= \frac{e^{-w/2}}{4} \int_0^w \; \mathrm{d}u \\ &= \frac{w\; e^{-w/2}}{4}, \end{aligned}$$

for $w > 0$. It then follows that

$$f_W(w) = \begin{cases} \dfrac{1}{4}\; w\; e^{-w/2} & \text{if } \; w > 0; \\ \\ 0 & \text{otherwise.} \end{cases}$$

This is the pdf for a $\chi^2(4)$ random variable. □

(e) *n degrees of freedom.* In this exercise we prove that if $W \sim \chi^2(n)$, then the pdf of W is given by

$$f_W(w) = \begin{cases} \dfrac{1}{\Gamma(n/2)\; 2^{n/2}}\; w^{\frac{n}{2}-1}\; e^{-w/2} & \text{if } \; w > 0; \\ \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where Γ denotes the Gamma function defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\; \mathrm{d}t \quad \text{for all real values of } z \text{ except } \; 0, -1, -2, -3, \ldots$$

Proof: We proceed by induction of n. Observe that when $n = 1$ the formula in (1) yields, for $w > 0$,

$$f_W(w) = \frac{1}{\Gamma(1/2)\; 2^{1/2}}\; w^{\frac{1}{2}-1}\; e^{-w/2} = \frac{1}{\sqrt{2\pi}}\;\frac{1}{\sqrt{x}}\; e^{-w/2},$$

which is the pdf for a $\chi^(1)$ random variable. Thus, the formula in (1) holds true for $n = 1$.

Next, assume that a $\chi^2(n)$ random variable has pdf given (1). We will show that if $W \sim \chi^2(n + 1)$, then its pdf is given by

$$f_W(w) = \begin{cases} \dfrac{1}{\Gamma((n+1)/2)\; 2^{(n+1)/2}}\; w^{\frac{n-1}{2}}\; e^{-w/2} & \text{if } w > 0; \\ \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

By the definition of a $\chi^2(n+1)$ random variable, we have that $W = X+Y$ where $X \sim \chi^2(n)$ and $Y \sim \chi^2(1)$ are independent random variables. It then follows that

$$f_W = f_X * f_Y$$

where

$$f_X(x) = \begin{cases} \dfrac{1}{\Gamma(n/2)\; 2^{n/2}}\; x^{\frac{n}{2}-1}\; e^{-x/2} & \text{if } x > 0; \\ \\ 0 & \text{otherwise.} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi}}\dfrac{1}{\sqrt{y}}\; e^{-y/2} & \text{if } y > 0; \\ \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, for $w > 0$,

$$\begin{aligned} f_W(w) &= \int_0^w \frac{1}{\Gamma(n/2)\; 2^{n/2}}\; u^{\frac{n}{2}-1}\; e^{-u/2}\; \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{w-u}}\; e^{-(w-u)/2}\; \mathrm{d}u \\ &= \frac{e^{-w/2}}{\Gamma(n/2)\sqrt{\pi}\; 2^{(n+1)/2}} \int_0^w \frac{u^{\frac{n}{2}-1}}{\sqrt{w-u}}\; \mathrm{d}u. \end{aligned}$$

Next, make the change of variables $t = u/w$; we then have that $u = wt$, $\mathrm{d}u = w\; \mathrm{d}t$ and

$$f_W(w) = \frac{w^{\frac{n-1}{2}} e^{-w/2}}{\Gamma(n/2)\sqrt{\pi}\; 2^{(n+1)/2}} \int_0^1 \frac{t^{\frac{n}{2}-1}}{\sqrt{1-t}}\; \mathrm{d}t.$$

Making a further change of variables $t = z^2$, so that $\mathrm{d}t = 2z\,\mathrm{d}z$, we obtain that

$$f_W(w) = \frac{2w^{\frac{n-1}{2}} e^{-w/2}}{\Gamma(n/2)\sqrt{\pi}\; 2^{(n+1)/2}} \int_0^1 \frac{z^{n-1}}{\sqrt{1-z^2}}\,\mathrm{d}z. \qquad (3)$$

It remains to evaluate the integrals

$$\int_0^1 \frac{z^{n-1}}{\sqrt{1-z^2}}\,\mathrm{d}z \qquad \text{for } n = 1, 2, 3, \ldots$$

We can evaluate these by making the trigonometric substitution $z = \sin\theta$ so that $\mathrm{d}z = \cos\theta\,\mathrm{d}\theta$ and

$$\int_0^1 \frac{z^{n-1}}{\sqrt{1-z^2}}\,\mathrm{d}z = \int_0^{\pi/2} \sin^{n-1}\theta\,\mathrm{d}\theta.$$

Looking up the last integral in a table of integrals we find that, if n is even and $n \geqslant 4$, then

$$\int_0^{\pi/2} \sin^{n-1}\theta\,\mathrm{d}\theta = \frac{1\cdot 3\cdot 5\cdots(n-2)}{2\cdot 4\cdot 6\cdots(n-1)},$$

which can be written in terms of the Gamma function as

$$\int_0^{\pi/2} \sin^{n-1}\theta\,\mathrm{d}\theta = \frac{2^{n-2}\left[\Gamma\left(\frac{n}{2}\right)\right]^2}{\Gamma(n)}. \qquad (4)$$

Note that this formula also works for $n = 2$.
Similarly, we obtain that for odd n with $n \geqslant 1$ that

$$\int_0^{\pi/2} \sin^{n-1}\theta\,\mathrm{d}\theta = \frac{\Gamma(n)}{2^{n-1}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2}\,\frac{\pi}{2}. \qquad (5)$$

Now, if n is odd and $n \geqslant 1$ we may substitute (5) into (3) to get

$$\begin{aligned} f_W(w) &= \frac{2w^{\frac{n-1}{2}} e^{-w/2}}{\Gamma(n/2)\sqrt{\pi}\; 2^{(n+1)/2}} \frac{\Gamma(n)}{2^{n-1}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2}\,\frac{\pi}{2} \\ &= \frac{w^{\frac{n-1}{2}} e^{-w/2}}{\Gamma(n/2)\; 2^{(n+1)/2}} \frac{\Gamma(n)\sqrt{\pi}}{2^{n-1}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2}. \end{aligned}$$

Now, by Problem 5 in Assignment 1, for odd n,

$$\Gamma\left(\frac{n}{2}\right) = \frac{\Gamma(n)\sqrt{\pi}}{2^{n-1}\;\Gamma\left(\frac{n+1}{2}\right)}.$$

It the follows that

$$f_W(w) \quad = \quad \frac{w^{\frac{n-1}{2}} e^{-w/2}}{\Gamma\left(\frac{n+1}{2}\right)\ 2^{(n+1)/2}}$$

for $w > 0$, which is (2) for odd n.

Next, suppose that n is a positive, even integer. In this case we substitute (4) into (3) and get

$$f_W(w) \quad = \quad \frac{2w^{\frac{n-1}{2}} e^{-w/2}}{\Gamma(n/2)\sqrt{\pi}\ 2^{(n+1)/2}}\ \frac{2^{n-2}\left[\Gamma\left(\frac{n}{2}\right)\right]^2}{\Gamma(n)}$$

or

$$f_W(w) \quad = \quad \frac{w^{\frac{n-1}{2}} e^{-w/2}}{2^{(n+1)/2}}\ \frac{2^{n-1}\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\ \Gamma(n)} \tag{6}$$

Now, since n is even, $n+1$ is odd, so that by by Problem 5 in Assignment 1 again, we get that

$$\Gamma\left(\frac{n+1}{2}\right) = \frac{\Gamma(n+1)\sqrt{\pi}}{2^n\ \Gamma\left(\frac{n+2}{2}\right)} = \frac{n\Gamma(n)\sqrt{\pi}}{2^n\ \frac{n}{2}\Gamma\left(\frac{n}{2}\right)},$$

from which we get that

$$\frac{2^{n-1}\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\ \Gamma(n)} = \frac{1}{\Gamma\left(\frac{n+1}{2}\right)}.$$

Substituting this into (6) yields

$$f_W(w) \quad = \quad \frac{w^{\frac{n-1}{2}} e^{-w/2}}{\Gamma\left(\frac{n+1}{2}\right)\ 2^{(n+1)/2}}$$

for $w > 0$, which is (2) for even n. This completes inductive step and the proof is now complete. That is, if $W \sim \chi^2(n)$ then the pdf of W is given by

$$f_W(w) = \begin{cases} \dfrac{1}{\Gamma(n/2)\ 2^{n/2}}\ w^{\frac{n}{2}-1}\ e^{-w/2} & \text{if } w > 0; \\ \\ 0 & \text{otherwise,} \end{cases}$$

for $n = 1, 2, 3, \ldots$ □

3.

4. *The t distribution.* Let $Z \sim \text{normal}(0,1)$ and $X \sim \chi^2(n-1)$ be independent random variables. Define
$$T = \frac{Z}{\sqrt{X/(n-1)}}.$$
Give the pdf of the random variable T.

***Solution**:* We first compute the cdf, F_T, of T; namely,

$$\begin{aligned} F_T(t) &= P(T \leqslant t) \\ &= P\left(\frac{Z}{\sqrt{X/(n-1)}} \leqslant t\right) \\ &= \iint_R f_{(X,Z)}(x,z)\ \mathrm{d}x\ \mathrm{d}z, \end{aligned}$$

where R_t is the region in the xz–plane given by
$$R_t = \{(x,z) \in \mathbb{R}^2 \mid z < t\sqrt{x/(n-1)}, x > 0\},$$
and the joint distribution, $f_{(X,Z)}$, of X and Z is given by
$$f_{(X,Z)}(x,z) = f_X(x) \cdot f_Z(z) \quad \text{for} \quad x > 0 \text{ and } \quad z \in \mathbb{R},$$
because X and Z are assumed to be independent. Furthermore,
$$f_X(x) = \begin{cases} \dfrac{1}{\Gamma\left(\frac{n-1}{2}\right)\ 2^{(n-1)/2}}\ x^{\frac{n-1}{2}-1}\ e^{-x/2} & \text{if } \ x > 0; \\ \\ 0 & \text{otherwise,} \end{cases}$$
and
$$f_Z(z) = \frac{1}{\sqrt{2\pi}}\ e^{-z^2/2}, \quad \text{for } -\infty < z < \infty.$$
We then have that
$$F_T(t) = \int_0^\infty \int_{-\infty}^{t\sqrt{x/(n-1)}} \frac{x^{\frac{n-3}{2}}\ e^{-(x+z^2)/2}}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}\ 2^{\frac{n}{2}}}\ \mathrm{d}z\ \mathrm{d}x.$$
Next, make the change of variables
$$\begin{aligned} u &= x \\ v &= \frac{z}{\sqrt{x/(n-1)}}, \end{aligned}$$

so that

$$\begin{aligned} x &= u \\ z &= v\sqrt{u/(n-1)}. \end{aligned}$$

Consequently,

$$F_T(t) = \frac{1}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}\, 2^{\frac{n}{2}}} \int_{-\infty}^{t}\int_{0}^{\infty} u^{\frac{n-3}{2}}\, e^{-(u+uv^2/(n-1))/2} \left|\frac{\partial(x,z)}{\partial(u,v)}\right| \,\mathrm{d}u\,\mathrm{d}v,$$

where the Jacobian of the change of variables is

$$\begin{aligned} \frac{\partial(x,z)}{\partial(u,v)} &= \det\begin{pmatrix} 1 & 0 \\ v/2\sqrt{u}\sqrt{n-1} & u^{1/2}/\sqrt{n-1} \end{pmatrix} \\ &= \frac{u^{1/2}}{\sqrt{n-1}}. \end{aligned}$$

It then follows that

$$F_T(t) = \frac{1}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{(n-1)\pi}\, 2^{\frac{n}{2}}} \int_{-\infty}^{t}\int_{0}^{\infty} u^{\frac{n}{2}-1}\, e^{-(u+uv^2/(n-1))/2} \,\mathrm{d}u\,\mathrm{d}v.$$

Next, differentiate with respect to t and apply the Fundamental Theorem of Calculus to get

$$\begin{aligned} f_T(t) &= \frac{1}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{(n-1)\pi}\, 2^{\frac{n}{2}}} \int_{0}^{\infty} u^{\frac{n}{2}-1}\, e^{-(u+ut^2/(n-1))/2} \,\mathrm{d}u \\ &= \frac{1}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{(n-1)\pi}\, 2^{\frac{n}{2}}} \int_{0}^{\infty} u^{\frac{n}{2}-1}\, e^{-\left(1+\frac{t^2}{n-1}\right)u/2} \,\mathrm{d}u. \end{aligned}$$

Put $\alpha = \dfrac{n}{2}$ and $\beta = \dfrac{2}{1+\frac{t^2}{n-1}}$. Then,

$$\begin{aligned} f_T(t) &= \frac{1}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{(n-1)\pi}\, 2^{\alpha}} \int_{0}^{\infty} u^{\alpha-1}\, e^{-u/\beta} \,\mathrm{d}u \\ &= \frac{\Gamma(\alpha)\beta^{\alpha}}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{(n-1)\pi}\, 2^{\alpha}} \int_{0}^{\infty} \frac{u^{\alpha-1}\, e^{-u/\beta}}{\Gamma(\alpha)\beta^{\alpha}} \,\mathrm{d}u, \end{aligned}$$

where

$$f_U(u) = \begin{cases} \dfrac{u^{\alpha-1}\, e^{-u/\beta}}{\Gamma(\alpha)\beta^\alpha} & \text{if } u > 0 \\ \\ 0 & \text{if } u \leqslant 0 \end{cases}$$

is the pdf of a $\Gamma(\alpha, \beta)$ random variable (see Problem 5 in Assignment #3). We then have that

$$f_T(t) \;=\; \frac{\Gamma(\alpha)\beta^\alpha}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{(n-1)\pi}\; 2^\alpha} \qquad \text{for } t \in \mathbb{R}.$$

Using the definitions of α and β we obtain that

$$f_T(t) \;=\; \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{(n-1)\pi}} \cdot \frac{1}{\left(1 + \dfrac{t^2}{n-1}\right)^{n/2}} \qquad \text{for } t \in \mathbb{R}.$$

This is the pdf of a random variable with a t distribution with $n-1$ degrees of freedom. In general, a random variable, T, is said to have a t distribution with r degrees of freedom, for $r \geqslant 1$, if its pdf is given by

$$f_T(t) \;=\; \frac{\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\sqrt{r\pi}} \cdot \frac{1}{\left(1 + \dfrac{t^2}{r}\right)^{(r+1)/2}} \qquad \text{for } t \in \mathbb{R}.$$

We write $T \sim t(r)$. Thus, in this example we have seen that, if $Z \sim \text{norma}(0, 1)$ and $X \sim \chi^2(n-1)$, then

$$\frac{Z}{\sqrt{X/(n-1)}} \sim t(n-1).$$

□