Assignment #2

Due on Wednesday, September 21, 2011

Read Section 1.2 on *The Vector Space* $\mathbb{R}^n$ in Baxandall and Liebek's text (pp. 2–9).

Read Sections 2.3 on *The Dot Product and Euclidean Norm* in the class Lecture Notes (pp. 11–12).

Read Sections 2.4 on *Orthogonality and Projections* in the class Lecture Notes (pp. 13–18).

Read Section 2.5 on *The Cross Product* in the class Lecture Notes (pp. 18–25).

Do the following problems

1. The vectors $v_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ span a two–dimensional subspace in $\mathbb{R}^3$, in other words, a plane through the origin. Give two unit vectors which are orthogonal to each other, and which also span the plane.

2. Use an appropriate orthogonal projection to compute the shortest distance from the point $P(1, 1, 2)$ to the plane in $\mathbb{R}^3$ whose equation is
$$2x + 3y - z = 6.$$

3. The dual space of of $\mathbb{R}^n$, denoted $(\mathbb{R}^n)^*$, is the vector space of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}$.

   For a given $w \in \mathbb{R}^n$, define $T_w \colon \mathbb{R}^n \to \mathbb{R}$ by
$$T_w(v) = w \cdot v \quad \text{for all} \quad v \in \mathbb{R}^n.$$

   Show that T_w is an element of the dual of $\mathbb{R}^n$ for all $w \in \mathbb{R}^n$.

4. Prove that for every linear transformation, $T \colon \mathbb{R}^n \to \mathbb{R}$, there exists $w \in \mathbb{R}^n$ such that
$$T(v) = w \cdot v \quad \text{for every} \quad v \in \mathbb{R}^n.$$
   (*Hint:* See where T takes the standard basis $\{e_1, e_2, \ldots, e_n\}$ in $\mathbb{R}^n$.)

5. Let $u_1, u_2, \ldots, u_n$ be unit vectors in $\mathbb{R}^n$ which are mutually orthogonal; that is,
$$u_i \cdot u_j = 0 \quad \text{for} \quad i \neq j.$$
Prove that the set $\{u_1, u_2, \ldots, u_n\}$ is a basis for $\mathbb{R}^n$, and that, for any $v \in \mathbb{R}^n$,
$$v = \sum_{i=1}^{n} (v \cdot u_i)\; u_i.$$

6. Let u, v and w denote non–zero vectors in $\mathbb{R}^3$. Given that $u \cdot w = 0$, $u \cdot v = c$, where c is a real constant, and $u \times v = w$, find the components of v in each of the three mutually orthogonal directions: u, w and $u \times w$.

7. Prove that the cross product is non–associative; that is, find three vectors u, v and w in $\mathbb{R}^3$ such that $(u \times v) \times w \neq u \times (v \times w)$.

8. Let v and w denote vectors in $\mathbb{R}^3$, and $\mathbf{0}$ the zero–vector in $\mathbb{R}^3$.
   (a) Prove that if $v \times w = \mathbf{0}$ and $v \cdot w = 0$, then at least one of v or w must be the zero vector.
   (b) Prove that $v \cdot (v \times w) = 0$.

9. In this problem and the next, we derive the vector identity
$$u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$$
for any vectors u, v and w in $\mathbb{R}^3$.
   (a) Argue that $u \times (v \times w)$ lies in the span of v and w. Consequently, there exist scalars t and s such that
$$u \times (v \times w) = tv + sw$$
   (b) Show that $(u \cdot v)t + (u \cdot w)s = 0$.

10. Let u, v and w be as in the previous problem.
   (a) Use the results of the previous problem to conclude that there exists a scalar r such that
$$u \times (v \times w) = r[(u \cdot w)v - (u \cdot v)w].$$
   (b) By considering some simple examples, deduce that $r = 1$ in the previous identity