Assignment #4

Due on Wednesday, October 5, 2011

Read Section 4.2 on *Continuity and Limits* in Baxandall and Liebek's text (pp. 185–188).

Read Section 3.3 on *Continuous Functions* in the class Lecture Notes (pp. 29–39).

Background and Definitions

- *(Open Sets in $\mathbb{R}^n$)* A set $U \subseteq \mathbb{R}^n$ is said to be open if either $U = \emptyset$, or for every $u \in U$ there exists a positive number r such that $B_r(u) \subset U$.
- *(Continuous Function)* Let U denote an open subset of $\mathbb{R}^n$. A function $F\colon U \to \mathbb{R}^m$ is said to be continuous at $x \in U$ if and only if $\lim_{\|y-x\|\to 0} \|F(y) - F(x)\| = 0$.
- *(Image)* If $A \subseteq U$, the *image of A* under the map $F\colon U \to \mathbb{R}^m$, denoted by $F(A)$, is defined as the set $F(A) = \{y \in \mathbb{R}^m \mid y = F(x) \text{ for some } x \in A\}$.
- *(Pre–image)* If $B \subseteq \mathbb{R}^m$, the *pre–image of B* under the map $F\colon U \to \mathbb{R}^m$, denoted by $F^{-1}(B)$, is defined as the set $F^{-1}(B) = \{x \in U \mid F(x) \in B\}$.

  Note that $F^{-1}(B)$ is always defined even if F does not have an inverse map.
- *(Continuous Functions 2)* Let U denote an open subset of $\mathbb{R}^n$. A function $F\colon U \to \mathbb{R}^m$ is continuous on U if and only if, for every open subset V of $\mathbb{R}^m$, the pre–image of V under F, $F^{-1}(V)$ is open in $\mathbb{R}^n$.
- *(Composition of Continuous Functions)* Let U denote an open subset of $\mathbb{R}^n$ and Q an open subset of $\mathbb{R}^m$. Suppose that the maps $F\colon U \to \mathbb{R}^m$ and $G\colon Q \to \mathbb{R}^k$ are continuous on their respective domains and that $F(U) \subseteq Q$. Then, the composition $G \circ F\colon U \to \mathbb{R}^k$ is continuous on U.

Do the following problems

1. Let U denote an open subset of $\mathbb{R}^n$. Suppose that $f\colon U \to \mathbb{R}$ is a scalar field and $G\colon U \to \mathbb{R}^m$ is vector valued function.
   (a) Explain how the product fG is defined.
   (b) Prove that if both f and G are continuous on U, then the vector valued function fG is also continuous on U.

2. Let U be an open subset of $\mathbb{R}^2$. Let $f\colon U \to \mathbb{R}$ and $g\colon U \to \mathbb{R}$ be two scalar fields on U, and define $h\colon U \to \mathbb{R}$ by

$$h(x,y) = f(x,y)g(x,y) \quad \text{for all} \quad (x,y) \in U.$$

Prove that if both f and g are continuous on U, then so is h.

Suggestion: First prove that the function $G\colon \mathbb{R}^2 \to \mathbb{R}$, defined by $G(x,y) = xy$ for all $(x,y) \in \mathbb{R}^2$, is continuous. Then, let $F\colon U \to \mathbb{R}^2$ denote the map given by

$$F(x,y) = (f(x,y), g(x,y)) \quad \text{for all} \quad (x,y) \in U,$$

and observe that $h = G \circ F$.

3. Let $U = \mathbb{R}^n \backslash \{\mathbf{0}\} = \{v \in \mathbb{R}^n \mid v \neq \mathbf{0}\}$.

   (a) Prove that U is an open subset of $\mathbb{R}^n$.

   (b) Define $f\colon \mathbb{R}^n \to \mathbb{R}$ by

   $$f(v) = \frac{1}{\|v\|} \quad \text{for all} \quad v \in U.$$

   Prove that f is continuous on U.

   Suggestion: Note that the function, g, defined by

   $$g(t) = \frac{1}{t} \quad \text{for all } t \neq 0,$$

   is continuous for $t \neq 0$.

4. Let $I \subseteq \mathbb{R}$ be an open interval and $\sigma\colon I \to \mathbb{R}^n$ be continuous path in $\mathbb{R}^n$ satisfying $\sigma(t) \neq \mathbf{0}$ for all $t \in I$. Define the function $f\colon I \to \mathbb{R}$ by

$$f(t) = \frac{1}{\|\sigma(t)\|} \quad \text{for all} \quad t \in I.$$

Prove that f is continuous on I.

5. Let $f(x,y) = \dfrac{x^2}{\sqrt{x^2+y^2}}$, for $(x,y) \neq (0,0)$. Define $f(0,0)$ so that the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is continuous at $(0,0)$. Explain why f is continuous everywhere in $\mathbb{R}^2$.

6. Let $f(x,y) = \dfrac{x^2}{x^2+y^2}$, for $(x,y) \neq (0,0)$. Show that the function f cannot be defined at $(0,0)$ so as to make it a continuous function.

7. Let U denote an open subset of $\mathbb{R}^n$ and suppose that $U \neq \emptyset$. A function $F\colon U \to \mathbb{R}^m$ is said to satisfy a Lipschitz condition on U if and only if there exists a positive constant, K, such that

$$\|F(v) - F(u)\| \leqslant K\|v-u\|, \qquad \text{for all } u, v \in U.$$

   Prove that if $F\colon U \to \mathbb{R}^m$ satisfies a a Lipschitz condition on U, then F is continuous on U.

8. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{x+y}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

   Determine whether or not f is continuous at $(0,0)$.

9. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

   Determine whether or not f is continuous at $(0,0)$.

10. Let

$$f(x,y) = \frac{x-y}{x+y}, \quad x+y \neq 0.$$

   Can f be defined on the line $x+y=0$ so that it is continuous at some point on this line?