Review Problems for Exam 1

1. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the plane given by $4x - y - 3z = 12$.

2. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the line given by the parametric equations

$$\begin{cases} x &= -1 + 4t, \\ y &= -7t, \\ z &= 2 - t. \end{cases}$$

3. Compute the area of the triangle whose vertices in $\mathbb{R}^3$ are the points $(1, 1, 0)$, $(2, 0, 1)$ and $(0, 3, 1)$

4. Let v and w be two vectors in $\mathbb{R}^3$, and let λ be a scalar. Show that the area of the parallelogram determined by the vectors v and $w + \lambda v$ is the same as that determined by v and w.

5. Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$ and $P_{\widehat{u}}(v)$ denote the orthogonal projection of v along the direction of $\widehat{u}$ for any vector $v \in \mathbb{R}^n$. Use the Cauchy–Schwarz inequality to prove that the map

$$v \mapsto P_{\widehat{u}}(v) \quad \text{for all } \; v \in \mathbb{R}^n$$

is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

6. Define $f \colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$ Prove that f is continuous at $(0, 0)$.

7. Show that

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is not continuous at $(0, 0)$.

8. Determine the value of L that would make the function

$$f(x,y) = \begin{cases} x \ \sin\left(\dfrac{1}{y}\right) & \text{if } \ y \neq 0; \\ \\ L & \text{otherwise }, \end{cases}$$

continuous at $(0,0)$. Is $f\colon \mathbb{R}^2 \to \mathbb{R}$ continuous on $\mathbb{R}^2$? Justify your answer.

9. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ be the path given by

$$\sigma(t) = (2\cos t, \sin t), \qquad \text{for } t \in \mathbb{R}.$$

(a) Sketch the image of σ.

(b) Find a tangent vector to the path at $t = \pi/4$.

(c) Give the parametric equations to the tangent line to the path at $t = \pi/4$. Sketch the line.

10. Let I denote an open interval, and $\sigma\colon I \to \mathbb{R}^n$ and $\gamma\colon I \to \mathbb{R}^n$ be differentiable paths on I. Define $h(t) = \sigma(t) \cdot \gamma(t)$ for all $t \in \mathbb{R}$. Show that $h\colon I \to \mathbb{R}$ is differentiable on I and verify that

$$h'(t) = \sigma'(t) \cdot \gamma(t) + \sigma(t) \cdot \gamma'(t), \qquad \text{for all } t \in I.$$

11. Let I denote an open interval, and $\sigma\colon I \to \mathbb{R}^n$ be a differentiable path satisfying $\|\sigma(t)\| = c$, a constant, for all $t \in I$. Show that, at any $t \in I$, $\sigma(t)$ is orthogonal to a tangent vector to the path at that t.

12. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ be given by $\sigma(t) = (t^{1/3}, t)$ for all $t \in \mathbb{R}$. Show that σ is not differentiable at 0.