Review Problems for Final Exam

1. Let P_1 and P_2 denote two distinct points in $\mathbb{R}^3$. Let v_1 and v_2 denote two linearly independent vectors in $\mathbb{R}^3$. Let ℓ_1 denote the line through P_1 in the direction of v_1, and ℓ_2 denote the line through P_2 in the direction of v_2. Assuming that ℓ_1 and ℓ_2 do not meet, give a formula for computing the distance from ℓ_1 to ℓ_2.

2. In this problem, x and y denote vectors in $\mathbb{R}^n$.

   Let $g\colon \mathbb{R}^n \to \mathbb{R}$ given by $g(x) = \sin(\|x\|)$, for all$x \in \mathbb{R}^n$. Prove that g is continuous on $\mathbb{R}^n$.

3. Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$. For a fixed vector v in $\mathbb{R}^n$, define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(t) = \|v - t\widehat{u}\|^2$, for all $t \in \mathbb{R}$. Show that g is differentiable and compute $g'(t)$ for all $t \in \mathbb{R}$.

   For any $v \in \mathbb{R}^n$, give the point on the line spanned by $\widehat{u}$ which is the closest to v. Justify your answer.

4. Let f be a real valued function which is C^1 in an open interval containing the closed an bounded interval $[a, b]$. Define C to be the portion of the graph of f over $[a, b]$; that is,

   $$C = \{(x, y) \in \mathbb{R}^2 \mid y = f(x),\ a \leqslant x \leqslant b\}.$$

   (a) Give a parametrization for C and compute the arc length, $\ell(C)$, of C.

   (b) Compute the arc length along the graph of $y = \ln x$ from $x = 1$ to $x = 2$.

5. Consider the iterated integral $\displaystyle\int_0^1 \int_{x^2}^1 x\sqrt{1 - y^2}\ dydx$.

   (a) Identify the region of integration, R, for this integral and sketch it.

   (b) Change the order of integration in the iterated integral and evaluate the double integral $\displaystyle\int_R x\sqrt{1 - y^2}\ dxdy$.

6. What is the region R over which you integrate when evaluating the iterated integral

$$\int_1^2 \int_1^x \frac{x}{\sqrt{x^2+y^2}} \, \mathrm{d}y \, \mathrm{d}x?$$

Rewrite this as an iterated integral first with respect to x, then with respect to y. Evaluate this integral. Which order of integration is easier?

7. Let R denote the region in the xy–plane given by

$$R = \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1, \ x^2 \leqslant y \leqslant x\}.$$

Sketch a picture the region R and evaluate the line integral $\displaystyle\int_{\partial R} x^2 \, \mathrm{d}x - xy \, \mathrm{d}y$, where ∂R is the boundary of R traversed in the counterclockwise sense.

8. Let $f \colon \mathbb{R} \to \mathbb{R}$ denote a twice–differentiable real valued function and define

$$u(x,y) = f(r) \quad \text{where} \quad r = \sqrt{x^2+y^2} \quad \text{for all} \quad (x,y) \in \mathbb{R}^2.$$

   (a) Define the vector field $F(x,y) = \nabla u(x,y)$. Express F in terms of f' and r.

   (b) Recall that the divergence of a vector field $F = P\,\widehat{i} + Q\,\widehat{j}$ is the scalar field given by $\mathrm{div}F = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$. Express the divergence of the gradient of u, in terms of f', f'' and r.
   The expression $\mathrm{div}(\nabla u)$ is called the Laplacian of u, and is denoted by Δu or $\nabla^2 u$.